On the rationality and continuity of logarithmic growth filtration of solutions of p-adic differential equations

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December 14, 2016

Abstract

We study the asymptotic behavior of solutions of Frobenius equations defined over the ring of overconvergent series. As an application, we prove Chiarellotto-Tsuzuki's conjecture on the rationality and right continuity of Dwork's logarithmic growth filtrations associated to ordinary linear p-adic differential equations with Frobenius structures.

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1 Introduction

We consider an ordinary linear p-adic differential equation

$$Dy = \frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_0 y = 0,$$

whose coefficients are bounded on the p-adic open unit disc $|x| < 1$. We define its solution space by

$$\mathrm{Sol}(D) := \{y \in \mathbb{Q}_p[\![x]\!]; Dy = 0\}.$$

In her study of p-adic elliptic functions, Lutz proves that any solution y of $Dy = 0$ has a non-zero radius of convergence r ([Lut37, Théorème IV]). In the paper [Dwo73a], Dwork studies the asymptotic behavior of y near the boundary $|x| = r$ assuming that any solution of $Dy = 0$ converges in a common open disc $|x| < r$. For simplicity, we assume $r = 1$. The most general result in this viewpoint is that y has a logarithmic growth (log-growth) $n-1$, that is,

$$\sup_{|x|=\rho} |y(x)| = O((\log(1/\rho))^{1-n}) \text{ as } \rho \uparrow 1.$$

Dwork also defines the so-called special log-growth filtration of $\mathrm{Sol}(D)$ by

$$\mathrm{Sol}_\lambda(D) := \{y \in \mathrm{Sol}(D); \sup_{|x|=\rho} |y(x)| = O((\log(1/\rho))^{-\lambda}) \text{ as } \rho \uparrow 1\}.$$

We assume that the a_i's are rational functions over $\mathbb{Q}_p$. Over the p-adic field, a naïve analogue of analytic continuation fails. In particular, the existence of local solutions of $Dy = 0$ at the disc $|x - a| < 1$ for any a does not imply the existence of global solutions. Even the p-adic exponential series $e^x = 1 + x + 2^{-1}x^2 + \ldots$, which is a solution of $dy/dx = y$, has a radius of convergence $p^{-1/(p-1)}$. Hence, it is natural to ask how the special log-growth filtration varies from disc to disc. Assume $p \neq 2$. In [Dwo82], Dwork provides an answer to this question for the hypergeometric differential equation

$$Dy = x(1-x)\frac{d^2y}{dx^2} + (1-2x)\frac{dy}{dx} - \frac{1}{4}y = 0,$$

which arises from the Legendre family of elliptic curves over $\mathbb{F}_p$

$$E_x : z^2 = w(w-1)(w-x),\ x \neq 0, 1.$$

Owing to its geometric origin, the hypergeometric differential equation admits a Frobenius structure: let $\bar{a} \in \mathbb{F}_p \setminus \{0, 1\}$, and let $a \in \mathbb{Z}_p$ be a lift of $\bar{a}$. The Frobenius slopes of the solution space of $Dy = 0$ at the disc $|x - a| < 1$ are $0, 1$ if $E_{\bar{a}}$ is ordinary, and $1/2, 1/2$ if $E_{\bar{a}}$ is supersingular. Dwork proves that the special log-growth filtration at the disc $|x - a| < 1$ coincides with the Frobenius slope filtration at the disc $|x - a| < 1$.

In the last few decades, p-adic differential equations have been extensively studied from many perspectives. As for the existence of solutions, André, Kedlaya, and Mebkhout ([And02],[Ked04],[Meb02]) independently prove the p-adic local monodromy theorem, which asserts the quasi-unipotence of p-adic differential equations defined over the Robba ring with Frobenius structures. Additionally, several striking applications of p-adic differential equations emerge: for example, Berger relates a certain p-adic representation of the absolute Galois group of $\mathbb{Q}_p$ to a p-adic differential equation over the Robba ring; then, he proves Fontaine's p-adic monodromy conjecture by using the p-adic local monodromy theorem ([Ber02]).

However, Dwork's works on the log-growth of solutions of p-adic differential equations have been neglected for a long period until Chiarellotto and Tsuzuki drew attention to it in [CT09]. We briefly summarize some recent developments on this subject.

- In [CT09], Chiarellotto and Tsuzuki formulate a fundamental conjecture on the log-growth filtrations for p-adic differential equations with Frobenius structures (see Conjecture 3.3). Their conjecture is two-fold. The first part can be stated as follows:

  Conjecture A (Conjecture 3.3 (i)). *Let $Dy = 0$ be a p-adic differential equation with a Frobenius structure. Then, the breaks of the filtration $\mathrm{Sol}_\bullet(D)$ are rational and $\mathrm{Sol}_\lambda(D) = \cap_{\mu>\lambda}\mathrm{Sol}_\mu(D)$ for all $\lambda \in \mathbb{R}$.*

The second part is about a comparison of the log-growth filtration and the Frobenius slope filtration under a certain technical assumption, which is based on Dwork's work on the hypergeometric differential equation. They prove the conjecture in the rank 2 case in [CT09]. They also provide a complete answer to a generic version of their conjecture in [CT11].

- In [And08], André proves Dwork's conjecture on a specialization property for the log-growth filtration, which is an analogue of Grothendieck-Katz specialization theorem on Frobenius structure.
- In [Ked10], Kedlaya studies effective convergence bounds on the solutions of p-adic differential equations with nilpotent singularities, which allows the a_i's to have a pole at $x = 0$. Then, he proves a partial generalization of Chiarellotto-Tsuzuki's earlier works to p-adic differential equations with nilpotent singularities.

Our main result in this paper is

Main Theorem (Theorem 3.7 (i)). *Conjecture A is true.*

Under a certain technical assumption, we also prove the second part of Chiarellotto-Tsuzuki's conjecture (Theorem 3.7 (ii)).

Strategy of proof

We present the proof of the rationality of breaks of the filtration $\mathrm{Sol}_\bullet(D)$. Let $\mathbb{Q}_p[\![x]\!]_0 := \mathbb{Z}_p[\![x]\!][p^{-1}]$ be the ring of bounded functions on the open unit disc, and σ a $\mathbb{Q}_p$-algebra endomorphism of $\mathbb{Q}_p[\![x]\!]_0$ such that $\sigma(x) = x^p$. Instead of a naïve p-adic differential equation $Dy = 0$, we consider a finite free $\mathbb{Q}_p[\![x]\!]_0$-module M of rank n endowed with an action of d/dx. The existence of a Frobenius structure of $Dy = 0$ is equivalent to the existence of a σ-semi-linear structure φ on M compatible with ∇. In [CT09], Chiarellotto and Tsuzuki establish a standard method for studying the log-growth filtration associated to M as follows. We fix a cyclic vector e of M as a σ-module over the fraction field of $\mathbb{Q}_p[\![x]\!]_0$. Let $V(M)$ be the set of horizontal sections of M after tensoring with the ring of analytic functions over the open unit disc. Let $v \in V(M)$ be a Frobenius eigenvector, i.e., $\varphi(v) = \lambda v$ for some $\lambda \in \mathbb{Q}_p$. If we write v as a linear combination of $e, \varphi(e), \dots, \varphi^{n-1}(e)$, then the coefficient f of $\varphi^{n-1}(e)$ satisfies a certain Frobenius equation

$$b_n f^{\sigma^n} + b_{n-1} f^{\sigma^{n-1}} + \cdots + b_0 f = 0,\ b_i \in \mathbb{Q}_p[\![x]\!]_0.$$

Then, the rationality of breaks of $\mathrm{Sol}_\bullet(D)$ is reduced to the rationality of the log-growth of f, i.e., the existence of $\lambda \in \mathbb{Q}$ such that

$$\sup_{|x|=\rho} |f(x)| = O((\log(1/\rho))^{-\lambda}) \text{ as } \rho \uparrow 1$$

and

$$\sup_{|x|=\rho} |f(x)| \neq O((\log(1/\rho))^{-\mu}) \text{ as } \rho \uparrow 1$$

for any $\mu < \lambda$. The rationality of the log-growth of f is proved by Chiarellotto and Tsuzuki in [CT09] when $n = 2$, then by Nakagawa in [Nak13] when n is arbitrary under the assumption that the number of breaks of the Newton polygon of $b_n X^n + b_{n-1} X^{n-1} + \cdots + b_0$ as a polynomial over the Amice ring $\mathcal{E}$ is equal to n. Nakagawa's assumption is too strong since it is equivalent to assuming that the number of breaks of the Frobenius filtration of M tensored with $\mathcal{E}$ is equal to n. Unfortunately, a naïve attempt to generalize Nakagawa's result without the assumption on the Newton polygon seems to fail.

To overcome this difficulty, we carefully choose a cyclic vector e in § 5: by definition, the Newton polygon of $b_n X^n + b_{n-1} X^{n-1} + \cdots + b_0$ is the boundary of the lower convex hull of some set of points associated to the b_i's. Our requirement for e is that each plotted point belongs to the Newton polygon. The construction of e is performed after a certain base change which is described in Kedlaya's framework of analytic rings. By using our cyclic vector e, the corresponding Frobenius equation is defined over Kedlaya's ring. Hence, we need to introduce a notion of log-growth on Kedlaya's ring (§ 4). Then, we generalize Nakagawa's calculation in § 6. Finally, we obtain the rationality of the log-growth filtration of $V(M)$ in § 7.

2 Summary of notation

We summarize our notation in this paper. Basically, we adopt the notation in [CT11]. In the appendix, we have a diagram describing relations between various rings defined in the following.

2.1 Coefficient rings

p : a prime number.

K : a complete discrete valuation field of characteristic $(0,p)$.

$\mathcal{O}_K$: the integer ring of K.

k_K : the residue field of K.

π_K : a uniformizer of $\mathcal{O}_K$.

$|\cdot|$: the p-adic absolute value on K^{alg} associated to a valuation of K, normalized by $|p|=p^{-1}$.

q : a positive power of p.

$q^s\in\mathbb{Q}$: Let s be a rational number and write $s=a/b$ with relatively prime $a,b\in\mathbb{Z}$. The notation "$q^s\in\mathbb{Q}$" means that b divides $\log_p q$, and we put $q^s:=p^{a\log_p q/b}$.

σ : a q-Frobenius on $\mathcal{O}_K$, i.e., a local ring endomorphism of $\mathcal{O}_K$ such that $\sigma(a)\equiv a^q \mod \pi_K\mathcal{O}_K$.

K^σ : the inductive limit of K

$$K\xrightarrow{\sigma}K\xrightarrow{\sigma}\dots.$$

We regard K^σ as an extension of K. Then, K^σ is a Henselian discrete valuation field, whose value group coincides with the value group of K, with residue field $k_K^{p^{-\infty}}$.

$K^{\sigma,\mathrm{ur}}$: the completion of the maximal unramified extension of K^σ. Then, $K^{\sigma,\mathrm{ur}}$ is a complete discrete valuation field, whose value group coincides with the value group of K, with the residue field k_K^{alg}. Moreover, σ induces a q-Frobenius on $K^{\sigma,\mathrm{ur}}$.

2.2 Various rings of functions

In the appendix, we have a diagram of the rings mentioned in this paper including the following rings of functions.

x : an indeterminate.

$|\cdot|_0^{\mathrm{naive}}(\rho)$: the multiplicative map

$$K[\![x]\!]\to\mathbb{R}_{\geq 0}\cup\{\infty\};\sum_{n\in\mathbb{N}}a_nx^n\mapsto\sup_{n\in\mathbb{N}}|a_n|\rho^n$$

defined for $\rho\in[0,1]$.

$K\{x\}$: the K-algebra of analytic functions on the open unit disc $|x|<1$, i.e.,

$$K\{x\}:=\left\{\sum_{n\in\mathbb{N}}a_nx^n\in K[\![x]\!];|a_n|\rho^n\to 0\ (n\to\infty)\forall\rho\in[0,1)\right\}.$$

Note that $|\cdot|_0^{\mathrm{naive}}(\rho)$ defines a multiplicative non-archimedean norm on $K\{x\}$ if $\rho\neq 0$.

$K[\![x]\!]_\lambda$: the Banach K-subspace of power series of logarithmic growth (log-growth) λ in $K\{x\}$ for $\lambda\in\mathbb{R}_{\geq 0}$, i.e.,

$$\begin{aligned}K[\![x]\!]_\lambda:&=\left\{\sum_{n\in\mathbb{N}}a_nx^n\in K[\![x]\!];\sup_{n\in\mathbb{N}}|a_n|/(n+1)^\lambda<\infty\right\}\\&=\left\{f\in K\{x\};|f|_0^{\mathrm{naive}}(\rho)=O((\log(1/\rho))^{-\lambda})\text{ as }\rho\uparrow 1\right\},\end{aligned}$$

where the last equality follows from [And08, Lemma 2.2.1 (iv)]. Note that $K[\![x]\!]_0$ coincides with the ring of bounded functions on the open unit disc $|x| < 1$, i.e.,

$$K[\![x]\!]_0 = \mathcal{O}_K[\![x]\!][\pi_K^{-1}] = \left\{ \sum_{n\in\mathbb{N}} a_n x^n \in K[\![x]\!]; \sup_{n\in\mathbb{N}} |a_n| < \infty \right\}.$$

We define $K[\![x]\!]_\lambda := 0$ for $\lambda \in \mathbb{R}_{<0}$. Note that $K[\![x]\!]_\lambda$ is stable under the derivation d/dx.

$\mathcal{E}$: the fraction field of the p-adic completion of $\mathcal{O}_K[\![x]\!][x^{-1}]$, i.e.,

$$\mathcal{E} := \left\{ \sum_{n\in\mathbb{Z}} a_n x^n \in K[\![x, x^{-1}]\!]; \sup_{n\in\mathbb{Z}} |a_n| < \infty, |a_n| \to 0 \text{ as } n \to -\infty \right\}.$$

Note that $\mathcal{E}$ is canonically endowed with a norm which is an extension of $|\cdot|_0^{\mathrm{naive}}(1)$. Then, $(\mathcal{E}, |\cdot|_0^{\mathrm{naive}}(1))$ is a complete discrete valuation field of mixed characteristic $(0,p)$ with uniformizer π_K and residue field $k_K(\!(x)\!)$.

$\mathcal{E}^\dagger$: the ring of overconvergent power series in $\mathcal{E}$, i.e.,

$$\mathcal{E}^\dagger := \left\{ \sum_{n\in\mathbb{Z}} a_n x^n \in \mathcal{E}; |a_n|\rho^n \to 0 \ (n \to -\infty) \text{ for some } \rho \in (0,1) \right\}.$$

Note that $(\mathcal{E}^\dagger, |\cdot|_0^{\mathrm{naive}}(1))$ is a Henselian discrete valuation field whose completion is $\mathcal{E}$.

$\mathcal{R}$: the Robba ring with variable x and coefficient K, i.e.,

$$\mathcal{R} := \left\{ \sum_{n\in\mathbb{Z}} a_n x^n \in K[\![x, x^{-1}]\!]; |a_n|\rho^n \to 0 \ (n \to \pm\infty) \ \forall \rho \in (\rho_0, 1) \text{ for some } \rho_0 \in (0,1) \right\}.$$

σ : a q-Frobenius on $\mathcal{O}_K[\![x]\!]$, which is an extension of σ, defined by fixing $\sigma(x) = x^q \mod \mathfrak{m}_K\mathcal{O}_K[\![x]\!]$. Note that σ induces ring endomorphisms on $K[\![x]\!]$, $K\{x\}$, $\mathcal{E}^\dagger$, $\mathcal{E}$, and $\mathcal{R}$, and $K[\![x]\!]_\lambda$ is stable under σ ([Chr83, 4.6.4]).

$\mathcal{E}_t$: a copy of $\mathcal{E}$ in which x is replaced by another indeterminate t. As above, we regard $\mathcal{E}_t$ as a complete discrete valuation field where t is a p-adic unit. In the literature, t is called Dwork's generic point ([Ked10, Definition 9.7.1]).

$\mathcal{E}_t[\![X-t]\!]_0$: the ring of bounded functions on $|X-t| < 1$ with variable $X-t$ and coefficient $\mathcal{E}_t$. We endow $\mathcal{E}_t[\![X-t]\!]_0$ with $\mathcal{E}$-algebra structure by the K-algebra homomorphism

$$\tau : \mathcal{E} \to \mathcal{E}_t[\![X-t]\!]_0; f \mapsto \sum_{n\in\mathbb{N}} \frac{1}{n!} \left(\frac{d^n f}{dx^n} \right)\bigg|_{x=t} (X-t)^n.$$

Since $\tau(K) \subset \mathcal{E}_t$ and $\tau(x) = X$, τ is equivariant under the derivations d/dx and d/dX. We define a q-Frobenius on $\mathcal{E}_t[\![X-t]\!]_0$ by $\sigma|_{\mathcal{E}_t} = \sigma$ (by identifying t as x) and $\sigma(X-t) = \tau(\sigma(x)) - \sigma(x)|_{x=t}$. Then, τ is also σ-equivariant.

$\mathcal{E}_t[\![X-t]\!]_\lambda$: the Banach $\mathcal{E}_t$-subspace of power series of log-growth λ in $\mathcal{E}_t\{X-t\}$.

Let R be either $K[\![x]\!]_0$, $K\{x\}$, $\mathcal{E}^\dagger$, $\mathcal{E}$, or $\mathcal{R}$. We define $\Omega^1_R := R dx$ with a K-linear derivation $d : R \to \Omega^1_R; f \mapsto (df/dx)dx$. We also endow Ω^1_R with a semi-linear σ-action defined by $\sigma(dx) := d\sigma(x)$. For $\mathcal{E}_t[\![X-t]\!]_0$ and $\mathcal{E}_t\{X-t\}$, we also define a corresponding $\Omega^1_\bullet$ by replacing K and x by $\mathcal{E}_t$ and $X-t$, respectively.

2.3 Filtration and Newton polygon

Let V be a finite dimensional vector space over a field F. Let $V^\bullet = \{V^\lambda\}_{\lambda\in\mathbb{R}}$ be a decreasing filtration by subspaces of V. Then, we define

$$V^{\lambda-} := \bigcap_{\mu<\lambda} V^\mu,\ V^{\lambda+} := \bigcup_{\mu>\lambda} V^\mu.$$

We say that $\lambda \in \mathbb{R}$ is a break of $V^\bullet$ if $V^{\lambda-} \neq V^{\lambda+}$. We also define the multiplicity of λ as $\dim_F V^{\lambda-} - \dim_F V^{\lambda+}$. We say that $V^\bullet$ is rational if all breaks of $V^\bullet$ are rational. We say that $V^\bullet$ is right continuous if $V^\lambda = V^{\lambda+}$ for all $\lambda \in \mathbb{R}$. We say that $V^\bullet$ is exhaustive or separated if $\cup_{\lambda\in\mathbb{R}} V^\lambda = V$ or $\cap_{\lambda\in\mathbb{R}} V^\lambda = 0$, respectively.

Similarly, for an increasing filtration $V_\bullet = \{V_\lambda\}_{\lambda\in\mathbb{R}}$ of V, we define

$$V_{\lambda-} := \bigcup_{\mu<\lambda} V_\mu,\ V_{\lambda+} := \bigcap_{\mu>\lambda} V_\mu.$$

We also define a break, a rationality, and a right continuity of $V_\bullet$ by replacing superscripts by subscripts.

We define the Newton polygon of a filtration as follows ([CT09, 3.3]). Let $\{V^\lambda\}_{\lambda\in\mathbb{R}}$ (resp. $\{V_\lambda\}_{\lambda\in\mathbb{R}}$) be a decreasing (resp. increasing) filtration of V. Let $\lambda_1 < \cdots < \lambda_n$ be the breaks of the filtration $V^\bullet$ (resp. $V_\bullet$) with multiplicities $m_1, \ldots, m_n$. We define the Newton polygon of $V^\bullet$ (resp. $V_\bullet$) as the piecewise linear function in the xy-plane whose left endpoint is $(0, 0)$, with slopes $\lambda_1, \ldots, \lambda_n$ whose projections to the x-axis have lengths $m_1, \ldots, m_n$.

3 Chiarellotto-Tsuzuki's conjectures and main theorem

We first recall the definition of (σ, ∇)-modules over $K[\![x]\!]_0$ and $\mathcal{E}$. Then, we recall the definition of the log-growth filtrations for (σ, ∇)-modules over $K[\![x]\!]_0$ and $\mathcal{E}$, and recall Chiarellotto-Tsuzuki's conjectures. After recalling known results on the conjectures, we state our main results. Our basic references are [CT09], [CT11], and [Ked10].

3.1 σ-modules

Let R be a commutative ring with a ring endomorphism δ. We denote $\delta(r)$ by r^δ if no confusion arises. A δ-module M is a finite free R-module M endowed with an R-linear isomorphism $\varphi : \delta^*M := R\otimes_{\delta,R} M \to M$. We can view M as a left module over the twisted polynomial ring $R\{\delta\}$ ([Ked10, 14.2.1]). If we regard φ as a δ-linear endomorphism of M, then (M, φ^n) for $n \in \mathbb{N}$ is a δ^n-module over R. For $\alpha \in R^\times$, $(M, \alpha\varphi)$ is also a δ-module over R.

Let M be a σ-module over K (K might be $\mathcal{E}$). We recall the Frobenius slope filtration of M ([CT09, § 2]). We say that M is étale if there exists an $\mathcal{O}_K$-lattice $\mathfrak{M}$ of M such that $\varphi(\mathfrak{M}) \subset \mathfrak{M}$ and $\varphi(\mathfrak{M})$ generates $\mathfrak{M}$. We say that M is pure of slope $\lambda \in \mathbb{R}$ if there exists $n \in \mathbb{N}_{>0}$ and $\alpha \in K$ such that $\log_{q^n} |\alpha| = -\lambda$ and $(M, \alpha^{-1}\varphi^n)$ is étale ([CT09, 2.1]). For a σ-module M over K, there exists a unique increasing filtration $\{S_\lambda(M)\}_{\lambda\in\mathbb{R}}$, called the slope filtration, of M such that $S_\lambda(M)/S_{\lambda-}(M)$ is pure of slope λ. We call the breaks of $S_\bullet(M)$ the Frobenius slopes of M. The following are some basic properties of the slope filtration:

- The slope filtration of M is exhaustive, separated, and right continuous.
- The Frobenius slopes of M are rational.
- The slope filtration of (M, φ^n) is independent of the choice of $n \in \mathbb{N}_{>0}$.

Assume that k_K is algebraically closed. Then, any short exact sequence of σ-modules splits ([Ked10, 14.3.4, 14.6.6]). Moreover, let M be a σ-module over K such that $q^\lambda \in \mathbb{Q}$ for any Frobenius slope λ of M. Then, M admits a basis consisting of elements of the form $\varphi(v) = q^\lambda v$ ([Ked10, 14.6.4]); we call v a Frobenius eigenvector of slope λ. In this situation, for any σ-submodules M' and M'' of M, we have $M' \subset M''$ if and only if any Frobenius eigenvector v of M' belongs to M''.

3.2 Log-growth filtration

Let R be either $K[\![x]\!]_0$ (K might be $\mathcal{E}$), $\mathcal{E}^\dagger$, $\mathcal{E}$, or $\mathcal{R}$. A ∇-module over R is a finite free R-module M endowed with a connection, i.e., a K-linear map

$$\nabla : M \to M \otimes_R \Omega^1_R = M dx$$

satisfying

$$\nabla(am) = m \otimes da + a\nabla(m)$$

for $a \in R$ and $m \in M$. A (σ, ∇)-module over R is a σ-module (M, φ) over R with a connection ∇ such that the following diagram is commutative:

$$\begin{tikzcd} M \arrow[r, "\nabla"] \arrow[d, "\varphi"] & M \otimes_R \Omega^1_R \arrow[d, "\varphi\otimes\sigma"] \\ M \arrow[r, "\nabla"] & M \otimes_R \Omega^1_R. \end{tikzcd}$$

(I) Special log-growth filtration ([CT09, 4.2])

Let M be a (σ, ∇)-module of rank n over $K[\![x]\!]_0$. We define the space of horizontal sections of M by

$$V(M) := (M \otimes_{K[\![x]\!]_0} K\{x\})^{\nabla=0}$$

and define the space of solutions of M by

$$\begin{aligned} \mathrm{Sol}(M) &:= \mathrm{Hom}_{K[\![x]\!]_0}(M, K\{x\})^{\nabla=0} \\ &:= \{f \in \mathrm{Hom}_{K[\![x]\!]_0}(M, K\{x\}); d(f(m)) = (f \otimes \mathrm{id})(\nabla(m)) \forall m \in M\}. \text{ (see [Ked10, } p. \text{ 82])} \end{aligned}$$

Both $V(M)$ and $\mathrm{Sol}(M)$ are known to be K-vector spaces of dimension n, and there exists a perfect pairing

$$V(M) \otimes_K \mathrm{Sol}(M) \to K$$

induced by the canonical pairing $M \otimes_{K[\![x]\!]_0} M^\vee \to K[\![x]\!]_0$, where $M^\vee$ denotes the dual of M. For $\lambda \in \mathbb{R}$, we define

$$\mathrm{Sol}_\lambda(M) := \mathrm{Hom}_{K[\![x]\!]_0}(M, K[\![x]\!]_\lambda) \cap \mathrm{Sol}(M),$$

which induces an increasing filtration of $\mathrm{Sol}(M)$. We say that M is solvable in $K[\![x]\!]_\lambda$ if $\dim_K \mathrm{Sol}_\lambda(M) = n$. We define

$$V(M)^\lambda := \mathrm{Sol}_\lambda(M)^\perp,$$

where $(\cdot)^\perp$ denotes the orthogonal space with respect to the above pairing. We call the decreasing filtration $\{V(M)^\lambda\}_\lambda$ the special log-growth filtration of M. Note that $\mathrm{Sol}_\bullet(M)$ and $V(M)^\bullet$ are exhaustive and separated. Moreover, $V(M)^\lambda$ (resp. $\mathrm{Sol}_\lambda(M)$) is a σ-submodule of $V(M)$ (resp. $\mathrm{Sol}(M)$) ([CT09, 4.8]).

Example. Let

$$Dy = \frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_0 y = 0,\ a_i \in K[\![x]\!]_0$$

be an ordinary linear p-adic differential equation. As in the introduction, we define

$$\mathrm{Sol}(D) := \{y \in K[\![x]\!]; Dy = 0\} \supset \mathrm{Sol}_\lambda(D) := \{y \in K[\![x]\!]_\lambda; Dy = 0\}.$$

We define a ∇-module $M := K[\![x]\!]_0 e_0 \oplus \cdots \oplus K[\![x]\!]_0 e_{n-1}$ by

$$\nabla(e_i) = \begin{cases} e_{i+1}dx & \text{if } 0 \le i \le n-2 \\ -(a_{n-1}e_{n-1} + \cdots + a_0 e_0)dx & \text{if } i = n-1. \end{cases}$$

Then, we have the canonical isomorphism

$$\mathrm{Sol}(M) \to \mathrm{Sol}(D); f \mapsto f(e_0),$$

under which we have

$$\mathrm{Sol}_\lambda(M) = \mathrm{Sol}_\lambda(D).$$

(II) Generic log-growth filtration ([CT09, § 4.1])

Let M be a (σ, ∇)-module over $\mathcal{E}$. We denote by $\tau^* M$ the pull-back of M under $\tau : \mathcal{E} \to \mathcal{E}_t[\![X-t]\!]_0$, which is a (σ, ∇)-module over $\mathcal{E}_t[\![X-t]\!]_0$. By a theorem of Robba, there exists a unique (σ, ∇)-submodule M^λ of M for $\lambda \in \mathbb{R}$ characterized as a minimal (σ, ∇)-submodule of M such that $\tau^*(M/M^\lambda)$ is solvable in $\mathcal{E}_t[\![X-t]\!]_\lambda$ ([CT09, 4.1]). We call the decreasing filtration $\{M^\lambda\}_{\lambda \in \mathbb{R}}$ of M the log-growth filtration of M. Note that $M^\bullet$ is exhaustive and separated, and if $M \neq 0$, then $M^\lambda \neq M$ for $\lambda \in \mathbb{R}_{\geq 0}$.

There exists a dual version of the log-growth filtration: for $\lambda \in \mathbb{R}$, we set $M_\lambda := ((M^\vee)^\lambda)^\perp$, where $(\cdot)^\perp$ denotes the orthogonal space with respect to the canonical pairing $M \otimes_{\mathcal{E}} M^\vee \to \mathcal{E}$. Then, M_λ is a maximal (σ, ∇)-submodule of M such that $\tau^* M_\lambda$ is solvable in $\mathcal{E}_t[\![X-t]\!]_\lambda$. Note that if $M \neq 0$, then $M_\lambda \neq 0$ for $\lambda \in \mathbb{R}_{\geq 0}$ by $(M^\vee)^\lambda \neq M^\vee$.

Note that the Frobenius slope filtration of M is stable under the action of ∇ ([CT09, 6.2]).

Definition 3.1. Let M be a (σ, ∇)-module over $K[\![x]\!]_0$.

(i) The Frobenius slope filtration $S_\bullet(V(M))$ of $V(M)$ is called the special Frobenius filtration of M ([CT09, 6.7]). We call a Frobenius slope of $V(M)$ a special Frobenius slope of M.

(ii) We put $M_{\mathcal{E}} := \mathcal{E} \otimes_{K[\![x]\!]_0} M$, which is a (σ, ∇)-module over $\mathcal{E}$. The Frobenius slope filtration $S_\bullet(M_{\mathcal{E}})$ of $M_{\mathcal{E}}$ is called the generic Frobenius filtration of M ([CT09, 6.1]). We call a Frobenius slope of $M_{\mathcal{E}}$ a generic Frobenius slope of M.

3.3 Chiarellotto-Tsuzuki's conjectures

In [Dwo73b, Concluding Remarks], Dwork observes that the log-growth and Frobenius slope filtrations can be compared. To formulate conjectures based on his observation, Chiarellotto and Tsuzuki introduce the following technical conditions:

Definition 3.2. (i) ([CT11, 6.1]) Let M be a (σ, ∇)-module over $\mathcal{E}$. We say that M is pure of bounded quotient (PBQ for short) if M/M^0 is pure as a σ-module.

(ii) ([CT11, 5.1]) Let M be a (σ, ∇)-module over $K[\![x]\!]_0$. We say that M is PBQ if $M_{\mathcal{E}}$ is PBQ. We say that M is horizontal of bounded quotient (HBQ for short) if there exists a quotient $\overline{M}$ of M as a (σ, ∇)-module such that there exists a canonical isomorphism $\overline{M}_{\mathcal{E}} \cong M_{\mathcal{E}}/M^0_{\mathcal{E}}$. Finally, we say that M is horizontally pure of bounded quotient (HPBQ for short) if M is PBQ and HBQ.

The following conjectures are first formulated by Chiarellotto and Tsuzuki in [CT09, § 6.4]. In this paper, we use the equivalent forms in [CT11].

Conjecture 3.3 (the conjecture $\mathbf{LGF}_{K[\![x]\!]_0}$ ([CT11, 2.5]))**.** *Let M be a (σ, ∇)-module over $K[\![x]\!]_0$.*

(i) The special log-growth filtration of M is rational and right continuous.

(ii) Let $\lambda_{\max}$ be the highest Frobenius slope of $M_{\mathcal{E}}$. If M is PBQ, then we have

$$V(M)^\lambda = (S_{\lambda - \lambda_{\max}}(V(M^\vee)))^\perp$$

for all $\lambda \in \mathbb{R}$. Here, $(\cdot)^\perp$ denotes the orthogonal space with respect to the canonical pairing $V(M) \otimes_K V(M^\vee) \to K$.

Conjecture 3.4 (the conjecture $\mathbf{LGF}_{\mathcal{E}}$ ([CT11, 2.4]))**.** *Let M be a (σ, ∇)-module over $\mathcal{E}$.*

(i) The log-growth filtration of M is rational and right continuous.

(ii) Let $\lambda_{\max}$ be the highest Frobenius slope of M. If M is PBQ, then we have

$$M^\lambda = (S_{\lambda - \lambda_{\max}}(M^\vee))^\perp$$

for all $\lambda \in \mathbb{R}$. Here, $(\cdot)^\perp$ denotes the orthogonal space with respect to the canonical pairing $M \otimes_{\mathcal{E}} M^\vee \to \mathcal{E}$.

To prove Chiarellotto-Tsuzuki's conjectures, we may assume that k_K is algebraically closed as remarked in [CT11, p. 42]. In the following, we recall known results on Chiarellotto-Tsuzuki's conjectures.

Theorem 3.5 ([CT11, Theorem 7.1, 7.2])**.** *The conjecture* $\mathbf{LGF}_{\mathcal{E}}$ *is true.*

Hence, the remaining part of Chiarellotto-Tsuzuki's conjectures is the conjecture $\mathbf{LGF}_{K[\![x]\!]_0}$.

Theorem 3.6. *Let M be a (σ,∇)-module of rank n over $K[\![x]\!]_0$.*

(i) ([CT09, Theorem 7.1 (2)]) The conjecture $\mathbf{LGF}_{K[\![x]\!]_0}$ *is true if $n\le 2$.*

(ii) ([CT11, Theorem 8.7]) The conjecture $\mathbf{LGF}_{K[\![x]\!]_0}$ *(i) is true if M is HBQ.*

(iii) ([CT09, Theorem 6.17]) For all $\lambda\in\mathbb{R}$, we have

$$V(M)^{\lambda}\subset(S_{\lambda-\lambda_{\max}}(V(M^{\vee})))^{\perp}.$$

(iv) ([CT11, Theorem 6.5]) The conjecture $\mathbf{LGF}_{K[\![x]\!]_0}$ *(ii) is true if M is HPBQ.*

(v) ([CT11, Proposition 7.3]) If the conjecture $\mathbf{LGF}_{K[\![x]\!]_0}$ *(ii) is true for an arbitrary M, then the conjecture* $\mathbf{LGF}_{K[\![x]\!]_0}$ *(i) is true for an arbitrary M.*

3.4 Main theorem

Our main result of this paper is

Theorem 3.7. *(i) The conjecture* $\mathbf{LGF}_{K[\![x]\!]_0}$ *(i) is true for an arbitrary M.*

(ii) The conjecture $\mathbf{LGF}_{K[\![x]\!]_0}$ *(ii) is true if the number of Frobenius slopes of $M_{\mathcal{E}}$ is less than or equal to 2.*

As mentioned in the introduction, we will study (σ,∇)-modules over $\mathcal{E}^{\dagger}$ rather than over $K[\![x]\!]_0$. Theorem 3.7 will follow from Theorem 4.19, which is a counterpart of Theorem 3.7 for (σ,∇)-modules over $\mathcal{E}^{\dagger}$.

4 Log-growth of analytic ring

In [Ked04] and [Ked05], Kedlaya provides functorial constructions of various analytic rings associated to a certain extension of $k_K(\!(x)\!)$. We recall some of his construction. After defining a notion of log-growth on Kedlaya's analytic rings, we develop a theory of log-growth filtrations for (σ,∇)-modules over $\mathcal{E}^{\dagger}$.

Notation 4.1. We set $\Gamma:=\mathcal{O}_{\mathcal{E}}\subset\Gamma^{\mathrm{alg}}:=\mathcal{O}_{\mathcal{E}^{\sigma,\mathrm{ur}}}$ for compatibility with the notation in the references. We denote the norm $|\cdot|_0^{\mathrm{naive}}(1)$ on $\Gamma[p^{-1}]$ by $|\cdot|_0(1)$, and extend $|\cdot|_0(1)$ to $\Gamma^{\mathrm{alg}}[p^{-1}]$.

Remark 4.2. The ring Γ^{alg} in [Ked04], which coincides with our Γ^{alg}, is different from that in [Ked05]: the latter contains our Γ^{alg}, but the residue field is the completion of $k_K(\!(x)\!)^{\mathrm{alg}}$. Fortunately, the definition of $\Gamma^{\mathrm{alg}}_{(\mathrm{an},)\mathrm{con}}$ comes out the same as mentioned in [Ked05, 2.4.13]. By regarding our $\Gamma^{\mathrm{alg}}_{(\mathrm{an},)\mathrm{con}}$ as a subring of $\Gamma^{\mathrm{alg}}_{(\mathrm{an},)\mathrm{con}}$ in [Ked05]. we may make (careful) use of the results of [Ked05].

4.1 Overconvergent rings

We define subrings Γ_{con} and $\Gamma^{\mathrm{alg}}_{\mathrm{con}}$ of Γ and Γ^{alg}, respectively, as follows: For $f\in\Gamma^{\mathrm{alg}}[p^{-1}]$, we have a unique expression

$$f=\sum_{i\gg-\infty}\pi_K^i[\bar{x}_i]$$

with $\bar{x}_i\in k_K(\!(x)\!)^{\mathrm{alg}}$, where $[\cdot]$ denotes Teichmüller lift. For $n\in\mathbb{N}$, we define the partial valuation $v_n:\Gamma^{\mathrm{alg}}[p^{-1}]\to\mathbb{R}\cup\{\infty\}$ by

$$v_n(f):=\min_{i\le n}\{v(\bar{x}_i)\},$$

where v denotes the non-archimedean valuation of $k_K(\!(x)\!)^{\mathrm{alg}}$ normalized by $v(x)=1$. For $r>0$, $n\in\mathbb{Z}$, and $f\in\Gamma^{\mathrm{alg}}[p^{-1}]$, we set

$$v_{n,r}(f)=rv_n(f)+n;$$

for $r=0$, we set $v_{n,r}(f)=n$ if $v_n(f)<\infty$ and $v_{n,r}(f)=\infty$ if $v_n(f)=\infty$. For $r\in\mathbb{R}_{\geq 0}$, we denote by Γ_r^{alg} the subring of $f\in\Gamma^{\mathrm{alg}}$ such that $\lim_{n\to\infty}v_{n,r}(f)=\infty$. On $\Gamma_r^{\mathrm{alg}}[p^{-1}]\backslash\{0\}$, we define the non-archimedean valuation

$$w_r(f):=\min_{n\in\mathbb{Z}}\{v_{n,r}(f)\}.$$

Note that $\Gamma_0^{\mathrm{alg}}=\Gamma^{\mathrm{alg}}$, and w_0 is a p-adic valuation on $\Gamma^{\mathrm{alg}}[p^{-1}]$ normalized by $w_0(\pi_K)=1$ ([Ked05, 2.1.11]). We define a multiplicative norm $|\cdot|_0(p^{-r}):=|\pi_K|^{w_r(\cdot)}$ on $\Gamma_r^{\mathrm{alg}}[p^{-1}]$. Define $\Gamma_{\mathrm{con}}^{\mathrm{alg}}:=\cup_{r\in\mathbb{R}_{>0}}\Gamma_r^{\mathrm{alg}}$. Since $\Gamma_r^{\mathrm{alg}}\subset\Gamma_s^{\mathrm{alg}}$ for $0<s\leq r$, we can define a value $|f|_0(\rho)\in\mathbb{R}_{\geq 0}$ of $f\in\Gamma_{\mathrm{con}}^{\mathrm{alg}}[p^{-1}]$ for $\rho\in(0,1)$ sufficiently close to 1 from the left. We define $\Gamma_{\mathrm{con}}:=\Gamma_{\mathrm{con}}^{\mathrm{alg}}\cap\Gamma$, and $\Gamma_r:=\Gamma_r^{\mathrm{alg}}\cap\Gamma$. Then, we have $\Gamma_{\mathrm{con}}=\mathcal{O}_{\mathcal{E}^\dagger}$ ([Ked05, 2.3.7]). Both $\Gamma_{\mathrm{con}}^{\mathrm{alg}}$ and Γ_{con} are Henselian discrete valuation rings ([Ked05, 2.1.12, 2.2.13]). Finally, note that $\Gamma_{\mathrm{con}}^{\mathrm{alg}}$, and hence, Γ_{con} is stable under σ and $|\sigma(\cdot)|_0(\rho)=|\cdot|_0(\rho^q)$ for $\rho\in(0,1)$.

Definition 4.3 ([Ked04, 3.5])**.** Let $f\in\Gamma_r^{\mathrm{alg}}[p^{-1}]$ be a non-zero element. We define the Newton polygon $\mathrm{NP}(f)$ of f as the boundary of the lower convex hull of the set of points $(v_n(f),n)$, minus any segments of slopes less than $-r$ from the left end and/or any segments of non-negative slope on the right end of the polygon. We define the slopes of f as the negatives of the slopes of $\mathrm{NP}(f)$. We also define the multiplicity of a slope $s\in(0,r]$ of f as the positive difference in y-coordinate between the endpoints of the segment of $\mathrm{NP}(f)$ of slope $-s$.

The following simple fact is one of the key points in this paper.

Lemma 4.4 (cf. [Nak13, Lemma 2.6])**.** *Let $f\in\Gamma_{\mathrm{con}}^{\mathrm{alg}}[p^{-1}]$. Then, there exists $\rho_0\in\mathbb{R}_{>0}$ and $a\in\mathbb{Q}$ such that*

$$|f|_0(\rho)=\rho^a|f|_0(1)\ \text{for all}\ \rho\in(\rho_0,1].$$

Proof. We may assume $f\neq 0$ and $f\in\Gamma_r^{\mathrm{alg}}[p^{-1}]$ for some $r>0$. Since the number of the slopes of $\mathrm{NP}(f)$ with non-zero multiplicities is finite by [Ked05, 2.4.6], we may assume that f has no slopes after choosing r sufficiently small. By [Ked05, 2.4.6] again, there exists a unique integer n such that $w_s(f)=v_{n,s}(f)$ for all $s\in[0,r]$. Then, we have $|f|_0(p^{-s})=(p^{-s})^{v_n(f)/e_K}|\pi_K|^n$ for any $s\in[0,r]$, where e_K is the absolute ramification index of K. By evaluating $s=0$, $|f|_0(1)=|\pi_K|^n$. Hence, we obtain the assertion for $\rho_0=p^{-r}$ and $a=v_n(f)/e_K$. $\square$

4.2 Log-growth filtration over $\Gamma_{\mathrm{con}}[p^{-1}]$

Throughout this section, let $\bullet$ denote either (blank) or alg. Let $\Gamma_{\mathrm{an},r}^\bullet$ be the Fréchet completion of the ring $\Gamma_r^\bullet[p^{-1}]$ with respect to the family of valuations $\{w_s\}_{s\in(0,r]}$ ([Ked04, 3.3]). We define $\Gamma_{\mathrm{an,con}}^\bullet:=\cup_{r\in\mathbb{R}_{>0}}\Gamma_{\mathrm{an},r}^\bullet$. Then, we have $\Gamma_{\mathrm{an,con}}=\mathcal{R}$, in particular, $\Gamma_{\mathrm{an,con}}$ contains $K\{x\}$. By continuity, $\Gamma_{\mathrm{an},r}^\bullet$ is endowed with a family of non-archimedean valuations induced by $\{v_n\}_{n\in\mathbb{Z}}$ and $\{w_s\}_{s\in(0,r]}$. In addition, the norm $|\cdot|_0(p^{-r})$ extends to $\Gamma_{\mathrm{an},r}^\bullet$. As before, we can define a value $|f|_0(\rho)\in\mathbb{R}_{\geq 0}$ of $f\in\Gamma_{\mathrm{an,con}}^\bullet$ for $\rho\in(0,1)$ sufficiently close to 1 from the left.

Remark 4.5. As mentioned above, we have $\Gamma_{\mathrm{con}}[p^{-1}]=\mathcal{E}^\dagger$ and $\Gamma_{\mathrm{an,con}}=\mathcal{R}$ as rings. However, the partial norms $|\cdot|_0(\rho)$ on $\Gamma_{\mathrm{an,con}}$ and $|\cdot|_0^{\mathrm{naive}}(\rho)$ on $\mathcal{R}$ coincide with each other only when ρ is sufficiently close to 1 ([Ked05, 2.3.5]). For this reason, we will distinguish $\Gamma_{\mathrm{con}}[p^{-1}]$ and $\Gamma_{\mathrm{an,con}}$ from $\mathcal{E}^\dagger$ and $\mathcal{R}$, respectively, as normed rings.

Definition 4.6 (Log-growth of analytic ring (cf. [Nak13, 2.8]))**.** For $\lambda\in\mathbb{R}$, we denote by $\mathrm{Fil}_\lambda\Gamma_{\mathrm{an,con}}^\bullet$ the subspace of $f\in\Gamma_{\mathrm{an,con}}^\bullet$ such that

$$|f|_0(\rho)=O((\log(1/\rho))^{-\lambda})\ \text{as}\ \rho\uparrow 1.$$

Lemma 4.7. *We have the following:*

(i) For a non-zero $f\in\Gamma_{\mathrm{an,con}}^\bullet$,

$$\liminf_{\rho\uparrow 1}|f|_0(\rho)>0.$$

(ii)

$$\mathrm{Fil}_0\Gamma_{\mathrm{an,con}}=\Gamma_{\mathrm{con}}[p^{-1}],\ \mathrm{Fil}_0\Gamma_{\mathrm{an,con}}^{\mathrm{alg}}\supset\Gamma_{\mathrm{con}}^{\mathrm{alg}}[p^{-1}].$$

(iii)
$$K\{x\} \cap \mathrm{Fil}_\lambda \Gamma_{\mathrm{an,con}} = K[\![x]\!]_\lambda \textit{ for } \lambda \in \mathbb{R}.$$

(iv)
$$\sigma(\mathrm{Fil}_\lambda \Gamma^\bullet_{\mathrm{an,con}}) \subset \mathrm{Fil}_\lambda \Gamma^\bullet_{\mathrm{an,con}} \textit{ for } \lambda \in \mathbb{R}.$$

(v)
$$\mathrm{Fil}_{\lambda_1} \Gamma^\bullet_{\mathrm{an,con}} \cdot \mathrm{Fil}_{\lambda_2} \Gamma^\bullet_{\mathrm{an,con}} \subset \mathrm{Fil}_{\lambda_1+\lambda_2} \Gamma^\bullet_{\mathrm{an,con}} \textit{ for } \lambda_1, \lambda_2 \in \mathbb{R}.$$

Proof. (i) We choose $r > 0$ sufficiently small such that $f \in \Gamma_{\mathrm{an},r}$. Then, we have $w_r(f) \neq \infty$ because $f \neq 0$. In particular, there exists $n \in \mathbb{Z}$ such that $v_n(f) \neq \infty$. By definition, $w_s(f) \leq sv_n(f) + n$ for all $s \in (0, r]$. Therefore, $\limsup_{s \downarrow 0} w_s(f) \leq n < \infty$, which implies the assertion.

(ii) By Lemma 4.4, we have only to prove that $f \in \mathrm{Fil}_0 \Gamma_{\mathrm{an,con}}$ belongs to $\Gamma_{\mathrm{con}}[p^{-1}]$. Since $|f|_0(\rho) = O(1)$ as $\rho \uparrow 1$, there exist a constant C and $r > 0$ such that $C < w_s(f)$ for all $s \in (0, r]$. If $v_n(f) < \infty$, then we have $C < n$ by taking the limit $s \downarrow 0$ in the inequality $w_s(f) \leq sv_n(f) + n$. Hence, we have $v_n(f) = \infty$ for all sufficiently small $n \in \mathbb{Z}$. If we take $l \in \mathbb{Z}$ such that $v_n(\pi_K^l f) = \infty$ for all $n < 0$, then we have $\pi_K^l f \in \Gamma_r$ by [Ked05, 2.5.6], i.e., $f \in \Gamma_{\mathrm{con}}[\pi_K^{-1}] = \Gamma_{\mathrm{con}}[p^{-1}]$.

(iii) It follows from the fact that for $f \in K\{x\}$, we have $|f|_0(\rho) = |f|_0^{\mathrm{naive}}(\rho)$ for ρ sufficiently close to 1 from the left ([Ked05, 2.3.5]).

(iv) It follows from $|\sigma(\cdot)|_0(\rho) = |\cdot|_0(\rho^q)$.

(v) The assertion follows from the multiplicativity of the norm $|\cdot|_0(\rho)$.

□

Note that $\mathrm{Fil}_\lambda \Gamma^\bullet_{\mathrm{an,con}} = 0$ for $\lambda \in \mathbb{R}_{<0}$ by Lemma 4.7 (i). In addition, Lemma 4.7 implies that $\mathrm{Fil}_\lambda \Gamma^\bullet_{\mathrm{an,con}}$ forms an increasing filtration of σ-stable $\Gamma^\bullet_{\mathrm{con}}[p^{-1}]$-subspaces of $\Gamma^\bullet_{\mathrm{an,con}}$.

Remark 4.8. In (i), the equality in the latter case does not hold. Indeed, there exists $f \in \Gamma^{\mathrm{alg}}_{\mathrm{an,con}}$ such that $v_n(f) = \infty$ for $n \in \mathbb{Z}_{<0}$, but $f \notin \Gamma^{\mathrm{alg}}_{\mathrm{con}}$ ([Ked05, 2.4.13]).

Definition 4.9 (A log extension of $\Gamma^\bullet_{\mathrm{an,con}}$ ([Ked04, 6.5]))**.** We set $\Gamma^\bullet_{\mathrm{log,an,con}} := \Gamma^\bullet_{\mathrm{an,con}}[\log x]$, where $\log x$ is an indeterminate. We can extend σ to $\Gamma^\bullet_{\mathrm{log,an,con}}$ as follows:

$$\sigma(\log x) := q \log x + \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} \left(\frac{\sigma(x)}{x^q} - 1\right)^i.$$

Moreover, we extend d/dx to $\Gamma_{\mathrm{log,an,con}} = \mathcal{R}[\log x]$ by

$$\frac{d}{dx}(\log x) = \frac{1}{x}.$$

We also define the notion of (σ, ∇)-modules over $\Gamma_{\mathrm{log,an,con}}$ by setting $R = \Gamma_{\mathrm{log,an,con}}$ and $\Omega^1_R = \Gamma_{\mathrm{log,an,con}} dx$ in § 3.2.

For $\rho \in (0, 1)$, we put $r := -\log_p \rho$ and extend $|\cdot|_0(\rho)$ to $\Gamma^\bullet_{\mathrm{an},r}[\log x]$ by

$$|\sum_{i \in \mathbb{N}} a_i (\log x)^i|_0(\rho) := \sup_{i \in \mathbb{N}} |a_i|_0(\rho) \cdot (\log(1/\rho))^{-i}.$$

Lemma 4.10. *The function $|\cdot|_0(\rho)$ is a multiplicative non-archimedean norm on the ring $\Gamma^\bullet_{\mathrm{an},r}[\log x]$.*

Proof. We have only to check the multiplicativity of $|\cdot|_0(\rho)$. Let $f = \sum_i a_i (\log x)^i, g = \sum_j b_j (\log x)^j \in \Gamma^\bullet_{\mathrm{an},r}[\log x]$. We have $|fg|_0(\rho) \leq |f|_0(\rho) \cdot |g|_0(\rho)$ by definition. We prove the converse. We may assume $f \neq 0$ and $g \neq 0$. Let i_0 (resp. j_0) be the minimum i (resp. j) such that $|f|_0(\rho) = |a_i|_0(\rho) \cdot (\log 1/\rho)^{-i}$ (resp. $|g|_0(\rho) = |b_j|_0(\rho) \cdot (\log 1/\rho)^{-j}$). For $i_1 < i_0$ and $j_0 \leq j_1$, we have

$$|a_{i_0}|_0(\rho) \cdot (\log(1/\rho))^{-i_0} > |a_{i_1}|_0(\rho) \cdot (\log(1/\rho))^{-i_1}, \ |b_{j_0}|_0(\rho) \cdot (\log(1/\rho))^{-j_0} \geq |b_{j_1}|_0(\rho) \cdot (\log(1/\rho))^{-j_1},$$

and hence, $|a_{i_0}b_{j_0}(\log x)^{i_0+j_0}|_0(\rho) > |a_{i_1}b_{j_1}(\log x)^{i_1+j_1}|_0(\rho)$. Similarly, we have

$$|a_{i_0}b_{j_0}(\log x)^{i_0+j_0}|_0(\rho) > |a_{i_1}b_{j_1}(\log x)^{i_1+j_1}|_0(\rho)$$

for $i_1 \geq i_0$ and $j_0 > j_1$. Therefore, we have

$$|fg|_0(\rho) \geq |\sum_{i+j=i_0+j_0} a_i b_j (\log x)^{i+j}|_0(\rho) = |f|_0(\rho) \cdot |g|_0(\rho).$$

□

Definition 4.11. We define a log-growth filtration of $\Gamma^{\bullet}_{\mathrm{log,an,con}}$ by

$$\mathrm{Fil}_{\lambda}\Gamma^{\bullet}_{\mathrm{log,an,con}} := \bigoplus_{i=0}^{\lfloor\lambda\rfloor} \mathrm{Fil}_{\lambda-i}\Gamma^{\bullet}_{\mathrm{an,con}} \cdot (\log x)^i$$

for $\lambda \in \mathbb{R}_{\geq 0}$ and $\mathrm{Fil}_{\lambda}\Gamma^{\bullet}_{\mathrm{log,an,con}} := 0$ for $\lambda \in \mathbb{R}_{<0}$. Here, $\lfloor\lambda\rfloor$ denotes the greatest integer less than or equal to λ. For $\lambda \in \mathbb{R}$, we say that $y \in \Gamma^{\bullet}_{\mathrm{log,an,con}}$ has a log-growth λ if $y \in \mathrm{Fil}_{\lambda}\Gamma^{\bullet}_{\mathrm{log,an,con}}$. Moreover, we say that f is bounded if f has a log-growth 0. For $\lambda \in \mathbb{R}_{>0}$, we also say that y is exactly of log-growth λ if $y \in \mathrm{Fil}_{\lambda}\Gamma^{\bullet}_{\mathrm{log,an,con}}$ and $y \notin \mathrm{Fil}_{\delta}\Gamma^{\bullet}_{\mathrm{log,an,con}}$ for any $0 \leq \delta < \lambda$ ([CT09, 1.1]).

Lemma 4.12. *(i) For $f \in \Gamma^{\bullet}_{\mathrm{log,an,con}}$ and $\lambda \in \mathbb{R}$, we have $f \in \mathrm{Fil}_{\lambda}\Gamma^{\bullet}_{\mathrm{log,an,con}}$ if and only if*

$$|f|_0(\rho) = O((\log(1/\rho))^{-\lambda}) \text{ as } \rho \uparrow 1.$$

(ii)

$$\sigma(\mathrm{Fil}_{\lambda}\Gamma^{\bullet}_{\mathrm{log,an,con}}) \subset \mathrm{Fil}_{\lambda}\Gamma^{\bullet}_{\mathrm{log,an,con}} \text{ for } \lambda \in \mathbb{R}.$$

(iii)

$$\mathrm{Fil}_{\lambda_1}\Gamma^{\bullet}_{\mathrm{log,an,con}} \cdot \mathrm{Fil}_{\lambda_2}\Gamma^{\bullet}_{\mathrm{log,an,con}} \subset \mathrm{Fil}_{\lambda_1+\lambda_2}\Gamma^{\bullet}_{\mathrm{log,an,con}} \text{ for } \lambda_1, \lambda_2 \in \mathbb{R}.$$

Proof. (i) The assertion follows from the definition.

(ii) The assertion follows from $\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i}\left(\frac{\sigma(x)}{x^q} - 1\right)^i \in \Gamma_{\mathrm{con}}[p^{-1}] = \mathrm{Fil}_0\Gamma_{\mathrm{an,con}}$ ([Ked04, 6.5]).

(iii) The assertion follows from Lemma 4.7 (v).

□

Definition 4.13 (Log-growth filtration)**.** Let M be a (σ, ∇)-module of rank n over $\Gamma_{\mathrm{con}}[p^{-1}]$. We set

$$\mathfrak{V}(M) := (\Gamma_{\mathrm{log,an,con}} \otimes_{\Gamma_{\mathrm{con}}[p^{-1}]} M)^{\nabla=0},$$

$$\mathfrak{Sol}(M) := \mathrm{Hom}_{\Gamma_{\mathrm{con}}[p^{-1}]}(M, \Gamma_{\mathrm{log,an,con}})^{\nabla=0} \cong \mathfrak{V}(M^{\vee}).$$

We say that M is solvable in $\Gamma_{\mathrm{log,an,con}}$ if $\dim_K \mathfrak{V}(M) = n$. In this case, we define

$$\mathfrak{Sol}_{\lambda}(M) := \mathrm{Hom}_{\Gamma_{\mathrm{con}}[p^{-1}]}(M, \mathrm{Fil}_{\lambda}\Gamma_{\mathrm{log,an,con}}) \cap \mathfrak{Sol}(M)$$

and

$$\mathfrak{V}(M)^{\lambda} := \mathfrak{Sol}_{\lambda}(M)^{\perp}$$

where $(\cdot)^{\perp}$ denotes the orthogonal space with respect to the canonical pairing $\mathfrak{V}(M) \otimes_K \mathfrak{Sol}(M) \to K$. We call the decreasing filtration $\{\mathfrak{V}(M)^{\lambda}\}_{\lambda}$ the special log-growth filtration of M.

Note that if M is a (σ, ∇)-module over $\Gamma_{\mathrm{con}}[p^{-1}]$ solvable in $\Gamma_{\mathrm{log,an,con}}$, then $\mathfrak{V}(M)$ is a σ-module over K by the injectivity of $\varphi : \mathfrak{V}(M) \to \mathfrak{V}(M)$. By Lemma 4.12 (ii), $\mathfrak{V}(M)^{\lambda}$ (resp. $\mathfrak{Sol}_{\lambda}(M)$) is a σ-submodule of $\mathfrak{V}(M)$ (resp. $\mathfrak{Sol}(M)$).

Example. We give an example of a ∇-module defined over $\Gamma_{\mathrm{con}}[p^{-1}]$, which is not necessary defined over $K[\![x]\!]_0$. Let $a = \sum_{n\in\mathbb{Z}} a_n x^n \in \Gamma_{\mathrm{con}}[p^{-1}]$ with $a_n \in K$ and we assume that there exists $\delta > 0$ such that

$$O(|a_{-n}|) = O(p^{-n\delta}) \text{ as } n \to \infty.$$

For example, assume $a_{-n} = 0$ for all $n \gg 0$. Let $M = \Gamma_{\mathrm{con}}[p^{-1}]e_1 \oplus \Gamma_{\mathrm{con}}[p^{-1}]e_2$ be the ∇-module of rank 2 over $\Gamma_{\mathrm{con}}[p^{-1}]$ defined by

$$\nabla(e_1, e_2) = (0, ae_1 dx).$$

We set

$$f = \sum_{n\neq -1} \frac{1}{n+1} a_n x^{n+1} + a_{-1}\log x,$$

which belongs to $\Gamma_{\mathrm{log,an,con}}$ as follows. For $n \geq 0$ and $\rho \in [0,1)$,

$$\left|\frac{1}{n+1} a_n x^{n+1}\right|_\rho \leq \frac{1}{n+1} \cdot \sup_{n\geq 0} |a_n| \cdot \rho^{n+1} \to 0 \text{ as } n \to \infty.$$

We fix $\rho_0 \in (p^{-\delta}, 1)$. For $\rho \in [\rho_0, 1)$ and $n \geq 2$,

$$\left|\frac{1}{-n+1} a_{-n} x^{-n+1}\right|_\rho \leq \frac{1}{n-1} \cdot \frac{|a_{-n}|}{p^{-n\delta}} \cdot \left(\frac{1}{p^\delta \rho}\right)^n \cdot \rho \leq \frac{1}{n-1} \cdot \frac{|a_{-n}|}{p^{-n\delta}} \cdot \left(\frac{1}{p^\delta \rho_0}\right)^n \to 0 \text{ as } n \to \infty.$$

Since the convergence is uniform with respect to ρ in the latter inequality, we have $f \in \mathrm{Fil}_\xi \Gamma_{\mathrm{log\,an,con}}$, where we define ξ as the log-growth of the power series $\sum_{n\geq 0} \frac{1}{n+1} a_n x^n$ if $a_{-1} = 0$, and $\xi = 1$ if $a_{-1} \neq 0$. Then, $\mathfrak{V}(M) := (\Gamma_{\mathrm{log,an,con}} \otimes_{\Gamma_{\mathrm{con}}[p^{-1}]} M)^{\nabla=0}$ has a basis $\{e_1, fe_1 - e_2\}$ and we have

$$\mathfrak{V}(M)^\lambda = \begin{cases} \mathfrak{V}(M) & \text{if } \lambda < 0 \\ Ke_1 & \text{if } 0 \leq \lambda < \xi \\ 0 & \text{if } \xi \leq \lambda. \end{cases}$$

Remark 4.14. We can define a special log-growth filtration for an arbitrary (σ, ∇)-module over $\Gamma_{\mathrm{con}}[p^{-1}]$ possibly after tensoring a suitable extension of $\Gamma_{\mathrm{con}}[p^{-1}]$ as follows. Let M be a (σ, ∇)-module over $\Gamma_{\mathrm{con}}[p^{-1}]$. Then, there exists a finite étale extension Γ^l/Γ, corresponding to a certain finite separable extension $l/k_K(\!(x)\!)$, such that $M' := \Gamma^l_{\mathrm{con}}[p^{-1}] \otimes_{\Gamma_{\mathrm{con}}[p^{-1}]} M$ is solvable in $\Gamma^l_{\mathrm{log,an,con}}$ by the log version of the p-adic local monodromy theorem ([Ked04, 6.13]). Similarly as above, we may define a special log-growth filtration of M'.

The log-growth filtrations are compatible with the base change $K[\![x]\!]_0 \to \mathcal{E}^\dagger = \Gamma_{\mathrm{con}}[p^{-1}]$:

Lemma 4.15. *Let M be a (σ, ∇)-module over $K[\![x]\!]_0$. Then, the (σ, ∇)-module $\Gamma_{\mathrm{con}}[p^{-1}] \otimes_{K[\![x]\!]_0} M$ over $\Gamma_{\mathrm{con}}[p^{-1}]$ is solvable in $\Gamma_{\mathrm{log,an,con}}$. Moreover, the canonical map*

$$\iota : V(M) \to \mathfrak{V}(\Gamma_{\mathrm{con}}[p^{-1}] \otimes_{K[\![x]\!]_0} M)$$

is an isomorphism, and preserves the Frobenius filtrations and the log-growth filtrations.

Proof. Since the natural inclusion $K\{x\} \to \Gamma_{\mathrm{log,an,con}}$ is compatible with Frobenius and differentials, $\Gamma_{\mathrm{con}}[p^{-1}] \otimes_{K[\![x]\!]_0} M$ is solvable in $\Gamma_{\mathrm{log,an,con}}$, and ι is an isomorphism of σ-modules over K. The rest of the assertion follows from $\mathrm{Fil}_\lambda \Gamma_{\mathrm{log,an,con}} \cap K\{x\} = \mathrm{Fil}_\lambda \Gamma_{\mathrm{an,con}} \cap K\{x\} = K[\![x]\!]_\lambda$ (Lemma 4.7 (iii)). □

4.3 Chiarellotto-Tsuzuki's conjecture over $\Gamma_{\mathrm{con}}[p^{-1}]$

We formulate an analogue of Theorem 3.7 for (σ, ∇)-modules over $\Gamma_{\mathrm{con}}[p^{-1}]$.

Assumption 4.16. In this section, we assume that k_K is algebraically closed for simplicity.

Definition 4.17. Let M be a (σ, ∇)-module over $\Gamma_{\mathrm{con}}[p^{-1}]$ solvable in $\Gamma_{\mathrm{log,an,con}}$.

(i) We call a Frobenius slope of $\mathfrak{V}(M)$ a special Frobenius slope of M.

(ii) We set $M_{\mathcal{E}} := \mathcal{E} \otimes_{\Gamma_{\mathrm{con}}[p^{-1}]} M$, which is a (σ, ∇)-module over $\mathcal{E}$. We call a Frobenius slope of $M_{\mathcal{E}}$ a generic Frobenius slope of M.

Proposition 4.18. *Let M be a (σ, ∇)-module over $\Gamma_{\mathrm{con}}[p^{-1}]$ solvable in $\Gamma_{\log,\mathrm{an},\mathrm{con}}$. Let $\lambda_{\max}$ be the highest Frobenius slope of $M_{\mathcal{E}}$.*

(i) (Analogue of [CT09, Theorem 6.17]) We have

$$\mathfrak{V}(M)^{\lambda} \subset (S_{\lambda-\lambda_{\max}}(\mathfrak{V}(M^{\vee})))^{\perp}$$

for all $\lambda \in \mathbb{R}$. Here, $(\cdot)^{\perp}$ denotes the orthogonal space with respect to the canonical pairing $\mathfrak{V}(M) \otimes_K \mathfrak{V}(M^{\vee}) \to K$.

(ii) If $M_{\mathcal{E}}$ is PBQ, then

$$\mathfrak{V}(M)^{0} = (S_{-\lambda_{\max}}(\mathfrak{V}(M^{\vee})))^{\perp}.$$

Theorem 4.19 (Generalization of Theorem 3.7)**.** *Let M be a (σ, ∇)-module over $\Gamma_{\mathrm{con}}[p^{-1}]$ solvable in $\Gamma_{\log,\mathrm{an},\mathrm{con}}$.*

(i) The special log-growth filtration of M is rational and right continuous .

(ii) Let $\lambda_{\max}$ be the highest Frobenius slope of $M_{\mathcal{E}}$. Assume that $M_{\mathcal{E}}$ is PBQ and the number of the Frobenius slopes of $M_{\mathcal{E}}$ is less than or equal to 2. Then,

$$\mathfrak{V}(M)^{\lambda} = (S_{\lambda-\lambda_{\max}}(\mathfrak{V}(M^{\vee})))^{\perp}$$

for all $\lambda \in \mathbb{R}$.

Recall that we may assume Assumption 4.16 to prove Conjecture 3.3 (§ 3.3). Hence, Theorem 3.7 follows from Theorem 4.19 by Lemma 4.15. The proofs of Proposition 4.18 and Theorem 4.19 will be given in § 7.

Remark 4.20. Obviously, one can formulate an analogue of Conjecture 3.3 (ii) for a (σ, ∇)-module over $\Gamma_{\mathrm{con}}[p^{-1}]$ such that $M_{\mathcal{E}}$ is PBQ.

4.4 Example: p-adic differential equations with nilpotent singularities

In [Ked10, § 18], Kedlaya studies effective bounds on the solutions of p-adic differential equations with nilpotent singularities. As an application, he proves a nilpotent singular analogue of Theorem 3.6 (iii) ([Ked10, Remark 18.4.4, Theorem 18.4.5]). In this subsection, we explain that a nilpotent singular analogue of Theorem 3.7 follows from Theorem 4.19.

In the following, we assume $\sigma(x) = x^q$. We define $\Omega^1_{K[\![x]\!]_0}(\log)$ as a σ-module of rank 1 over $K[\![x]\!]_0$ with basis dx/x such that $\sigma^*(1 \otimes dx/x) := q dx/x$. Let

$$d : K[\![x]\!]_0 \to \Omega^1_{K[\![x]\!]_0}(\log) = K[\![x]\!]_0 dx/x; f \mapsto x df/dx \cdot dx/x$$

be the canonical derivation on $K[\![x]\!]_0$. We define a log (σ, ∇)-module over $K[\![x]\!]_0$ similarly to § 3.2 by setting $R = K[\![x]\!]_0$ and $\Omega^1_R = \Omega^1_{K[\![x]\!]_0}(\log)$.

As in Definition 4.11, we define a log-growth filtration of $K[\![x]\!][\log x]$ as

$$K[\![x]\!][\log x]_{\lambda} := \bigoplus_{i=0}^{\lfloor \lambda \rfloor} K[\![x]\!]_{\lambda-i} (\log x)^i$$

for $\lambda \in \mathbb{R}_{\geq 0}$ and $K[\![x]\!][\log x]_{\lambda} := 0$ for $\lambda \in \mathbb{R}_{<0}$. For a log (σ, ∇)-module M over $K[\![x]\!]_0$, we define

$$V(M) := (K\{x\}[\log x] \otimes_{K[\![x]\!]_0} M)^{\nabla=0}.$$

By Dwork's trick, $V(M)$ is of dimension n ([Ked10, Corollary 17.2.4]). We define a special log-growth filtration $V(M)^{\bullet}$ of M as in § 4.2 by replacing $K[\![x]\!]_{\lambda}$ by $K[\![x]\!][\log x]_{\lambda}$.

Example. (i) A (σ,∇)-module over $K[\![x]\!]_0$ can be regarded as a log (σ,∇)-module over $K[\![x]\!]_0$ by identifying dx as $x\cdot dx/x$. The special log-growth filtration of M as a non-log or log (σ,∇)-module coincides with each other.

(ii) Let $M := K[\![x]\!]_0 e_1 \oplus K[\![x]\!]_0 e_2$ be the log (σ,∇)-module of rank 2 over $K[\![x]\!]_0$ defined by

$$\nabla(e_1,e_2) = (0, e_1 dx/x),\ \varphi(e_1,e_2) = (e_1, qe_2).$$

Then, $V(M)$ has a basis $\{e_1, -\log x\cdot e_1 + e_2\}$. Moreover, the Frobenius slopes of $V(M)$ are $0,1$, and we have

$$V(M)^\lambda = \begin{cases} V(M) & \text{if } \lambda<0 \\ Ke_1 & \text{if } 0\le\lambda<1 \\ 0 & \text{otherwise.}\end{cases}$$

Our main result in this subsection is

Theorem 4.21. *An analogue of Theorem 3.7 for log (σ,∇)-modules over $K[\![x]\!]_0$ holds.*

Proof. It follows from Theorem 4.19 thanks to Lemma 4.22 below. □

Lemma 4.22. *Let M be a log (σ,∇)-module over $K[\![x]\!]_0$. Then, the (σ,∇)-module $\Gamma_{\mathrm{con}}[p^{-1}]\otimes_{K[\![x]\!]_0} M$ over $\Gamma_{\mathrm{con}}[p^{-1}]$ is solvable in $\Gamma_{\log,\mathrm{an},\mathrm{con}}$. Moreover, the canonical map*

$$\iota : V(M) \to \mathfrak{V}(\Gamma_{\mathrm{con}}[p^{-1}]\otimes_{K[\![x]\!]_0} M)$$

is an isomorphism, and preserves the Frobenius filtrations and log-growth filtrations.

Proof. Similar to the proof of Lemma 4.15. □

5 Generic cyclic vector

In this section, we prove a key technical result in this paper concerned with a σ-module over $\Gamma^{\mathrm{alg}}_{\mathrm{con}}[p^{-1}]$.

Definition 5.1. Let R be either $\Gamma^{\mathrm{alg}}_{\mathrm{con}}[p^{-1}]$ or $\Gamma^{\mathrm{alg}}[p^{-1}]$.

(i) For $f(\sigma) = a_0 + a_1\sigma + \cdots + a_n\sigma^n \in R\{\sigma\}$ a twisted polynomial, we define the Newton polygon $\mathrm{NP}(f(\sigma))$ of $f(\sigma)$ as the boundary of the lower convex hull of the set of points

$$\{(i, -\log_q |a_i|_0(1)); 0\le i\le n\}.$$

A slope of $\mathrm{NP}(f(\sigma))$ is called a slope of $f(\sigma)$ (cf. [Ked10, 2.1.3]; note that the Newton polygon in [Ked10] is defined as the boundary of the lower convex hull of $\{(-i, -\log_q |a_i|_0(1)); 0\le i\le n\}$. Consequently, our slopes coincide with -1 times the slopes in [Ked10]). We consider the following condition $(*)$ on $f(\sigma)$:

$$(*):\ \text{each point } (i, -\log_q |a_i|_0(1)) \text{ belongs to } \mathrm{NP}(f(\sigma)).$$

(ii) Let M be a σ-module of rank n over R. When $R = \Gamma^{\mathrm{alg}}_{\mathrm{con}}[p^{-1}]$, we call a Frobenius slope (resp. the Newton polygon) of $\Gamma^{\mathrm{alg}}_{\mathrm{con}}[p^{-1}]\otimes_R M$ a generic Frobenius slope (resp. the generic Newton polygon) of M. We say that an element $e\in M$ is a cyclic vector if $e, \varphi(e), \dots, \varphi^{n-1}(e)$ is a basis of M over R. For a cyclic vector e, we have a unique relation

$$\varphi^n(e) = -(a_{n-1}\varphi^{n-1}(e) + \cdots + a_0 e)$$

with $a_i\in R$. We set $f_e(\sigma) := a_0 + a_1\sigma + \cdots + \sigma^n \in R\{\sigma\}$. Note that $\mathrm{NP}(f_e(\sigma))$ coincides with the (generic) Frobenius Newton polygon of $M^\vee$ ([Ked10, 14.5.7]).

We say that a cyclic vector $e\in M$ is generic if $f_e(\sigma)$ satisfies the condition (*).

Theorem 5.2. *Let M be a σ-module over $\Gamma^{\mathrm{alg}}[p^{-1}]$ (resp. $\Gamma^{\mathrm{alg}}_{\mathrm{con}}[p^{-1}]$). Assume $q^s\in\mathbb{Q}$ for any (resp. generic) Frobenius slope s of M. Then, there exists a generic cyclic vector of M.*

In the next subsection, we see that there exists a non-empty open subset U of M such that $v\in U$ is a generic cyclic vector. In this sense, there exist a number of cyclic vectors satisfying the condition (*). Therefore, the condition (*) is referred to as being generic.

5.1 Proof of Theorem 5.2

To prove Theorem 5.2, we first construct a generic cyclic vector over $\Gamma^{\mathrm{alg}}[p^{-1}]$. Then, we deform it to obtain a generic cyclic vector over $\Gamma^{\mathrm{alg}}_{\mathrm{con}}[p^{-1}]$. We first recall Kedlaya's algorithm to compute an annihilator of an element of a σ-module over $\Gamma^{\mathrm{alg}}[p^{-1}]$ ([Ked05, 5.2.4]).

Construction 5.3. Let $R := \Gamma^{\mathrm{alg}}[p^{-1}]$. Let M be a σ-module of rank n over R with Frobenius slopes $s_1 \leq \cdots \leq s_n$ with multiplicities. Assume $q^{s_i} \in \mathbb{Q}$ for all i (§ 3.1). Then, we can choose an R-basis $e_1, \dots, e_n$ of M such that $\varphi(e_i) = q^{s_i} e_i$ for all i. Fix $x_1 \dots, x_n \in R$ and set $v := x_1 e_1 + \cdots + x_n e_n$. We define $v_l \in M$ for $1 \leq l \leq n$ by induction on l. Set $v_1 := v$. Given v_l, write $v_l = x_{l,1} e_1 + \cdots + x_{l,n} e_n$ with $x_{l,i} \in R$ and define

$$b_l := \begin{cases} q^{s_l} \cdot \sigma(x_{l,l})/x_{l,l} & \text{if } x_{l,l} \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

and $v_{l+1} := (\varphi - b_l) v_l$. Then, we have $v_l \in Re_l + \cdots + Re_n$ and $(\varphi - b_n) \dots (\varphi - b_1) v = 0$. We write

$$(\sigma - b_n) \dots (\sigma - b_1) = \sigma^n + c_{n-1} \sigma^{n-1} + \cdots + c_0, \ c_i \in R$$

in $R\{\sigma\}$. By construction, we may regard $x_{l,i} = x_{l,i}(\boldsymbol{x})$, $b_l = b_l(\boldsymbol{x})$, and $c_i = c_i(\boldsymbol{x})$ as functions of $\boldsymbol{x} = (x_1, \dots, x_n) \in R^n$ with values in R. We also regard $v = v(\boldsymbol{x})$ as a function of $\boldsymbol{x}$ with values in M.

Lemma 5.4. *We retain the notation in Construction 5.3.*

(i) For $\boldsymbol{x} \in R^n$, $v(\boldsymbol{x})$ is a cyclic vector of M if and only if $x_{1,1}(\boldsymbol{x}) x_{2,2}(\boldsymbol{x}) \dots x_{n,n}(\boldsymbol{x}) \neq 0$.

(ii) For $\boldsymbol{x} \in R^n$, $v(\boldsymbol{x})$ is a generic cyclic vector of M if and only if $x_{1,1}(\boldsymbol{x}) x_{2,2}(\boldsymbol{x}) \dots x_{n,n}(\boldsymbol{x}) \neq 0$ and $-\log_q |c_i(\boldsymbol{x})|_0(1) = s_1 + \cdots + s_{n-i}$ for all i.

(iii) Let $\boldsymbol{x}^{(0)} \in R^n$. Assume that $b_1(\boldsymbol{x}^{(0)}), \dots, b_n(\boldsymbol{x}^{(0)})$ are all non-zero. Then, there exists an open neighborhood $U \subset R^n$ of $\boldsymbol{x}^{(0)}$ (with respect to the topology induced by $|\cdot|_0(1)$) such that all b_l and $x_{l,i}$ are continuous on U. In particular, all c_i are also continuous on U.

Proof. (i) By construction, there exists an upper triangular matrix T whose diagonals are $(1, \dots, 1)$ such that $(v_1, v_2, \dots, v_n) = (v, \varphi(v), \dots, \varphi^{n-1}(v)) T$. Since $\{x_{l,i}\}_{l,i}$ is an upper triangular matrix, we obtain the assertion.

(ii) It follows from (i) and the fact that the slopes of $\sigma^n + c_{n-1}\sigma^{n-1} + \cdots + c_0$ are $-s_n \leq \cdots \leq -s_1$ with multiplicities.

(iii) By induction on $l \in \{1, \dots, n\}$, we construct an open neighborhood $U_l \subset R^n$ of $\boldsymbol{x}^{(0)}$ such that $x_{l,1}, \dots, x_{l,n}$ and b_l are continuous on U_l, and $x_{l,l}$ is non-zero on U_l. Once we construct the U_l's, $U := U_1 \cap \cdots \cap U_n$ satisfies the desired condition. First, note that $x_{l,l}(\boldsymbol{x}^{(0)}) \neq 0$ for all l by assumption. The assertion is trivial for $l = 1$ by setting $U_1 := \{\boldsymbol{x} \in R^n; x_1(\boldsymbol{x}) \neq 0\}$. Given U_{l-1}, let $U'_l := U_{l-1} \cap \{\boldsymbol{x} \in R^n; x_{l-1,l-1}(\boldsymbol{x}) \neq 0\}$, which is an open neighborhood of $\boldsymbol{x}^{(0)}$. By the induction hypothesis, $x_{l,i} = \sigma(x_{l-1,i}) q^{s_i} - b_{l-1} x_{l-1,i}$ is continuous on U'_l. We set $U_l := U'_l \cap \{\boldsymbol{x} \in R^n; x_{l,l}(\boldsymbol{x}) \neq 0\}$. Then, $U_l \subset R^n$ is an open neighborhood of $\boldsymbol{x}^{(0)}$ on which b_l is continuous on U_l as desired. □

Lemma 5.5. *Let $s_1 \leq s_2 \leq \cdots \leq s_n$ be rational numbers such that $q^{s_i} \in \mathbb{Q}$. Then, the slopes of $f(\sigma) := (\sigma - q^{s_1} x) \dots (\sigma - q^{s_n} x) \in \Gamma[p^{-1}]\{\sigma\}$ are $-s_n \leq \cdots \leq -s_1$ with multiplicities. Moreover, $f(\sigma)$ satisfies the condition (*).*

Proof. We write

$$f(\sigma) = \sigma^n + a_{n-1}\sigma^{n-1} + \cdots + a_0, \ a_i \in \Gamma[p^{-1}].$$

Then, we have

$$a_{n-i} = \sum_{1 \leq j(1) < \cdots < j(i) \leq n} (-1)^i q^{s_{j(1)} + \cdots + s_{j(i)}} x^{q^{j(1)-1} + \cdots + q^{j(i)-1}}.$$

Hence, $-\log_q |a_{n-i}|_0(1) = s_1 + \cdots + s_i$, which implies the assertion. □

Proof of Theorem 5.2. We first consider the case where M is a σ-module over $R := \Gamma^{\mathrm{alg}}[p^{-1}]$. Let $s_1 \leq s_2 \leq \cdots \leq s_n$ be the Frobenius slopes of M with multiplicities. By Lemma 5.5, the slopes of the σ-module $M' := R\{\sigma\}/R\{\sigma\}(\sigma - q^{s_1}x)(\sigma - q^{s_2}x)\dots(\sigma - q^{s_n}x)$ are $s_1, \dots, s_n$ with multiplicities. Recall that σ-modules over R are classified by its slopes with multiplicities by Dieudonné-Manin theorem ([Ked10, 14.6.3]). Hence, there exists an isomorphism of σ-modules $M \cong M'$. By Lemma 5.5, $\bar{1} \in M'$ is a generic cyclic vector.

We consider the case where M is a σ-module over $\Gamma^{\mathrm{alg}}_{\mathrm{con}}[p^{-1}]$. We have only to prove that there exists $f \in M$ which is a generic cyclic vector of $M^{\mathrm{alg}} := \Gamma^{\mathrm{alg}}[p^{-1}] \otimes_{\Gamma^{\mathrm{alg}}_{\mathrm{con}}[p^{-1}]} M$. We apply Construction 5.3 to M^{alg}. We choose a generic cyclic vector e of M^{alg} and write $e = v(\boldsymbol{x}^{(0)})$ with $\boldsymbol{x}^{(0)} \in R^n$. By Lemma 5.4 (ii) and (iii), there exists an open neighborhood $U \subset R^n$ of $\boldsymbol{x}^{(0)}$ such that $v(\boldsymbol{x})$ is a generic cyclic vector of M^{alg} for all $\boldsymbol{x} \in U$. We choose a $\Gamma^{\mathrm{alg}}_{\mathrm{con}}[p^{-1}]$-basis $f_1, \dots, f_n$ of M. For $\boldsymbol{y} = (y_1, \dots, y_n) \in R^n$, we define $w(\boldsymbol{y}) := y_1 f_1 + \cdots + y_n f_n$. For $\boldsymbol{x} \in R^n$, there exists a unique $\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{x}) \in R^n$ such that $v(\boldsymbol{x}) = w(\boldsymbol{y})$, and the map $\boldsymbol{x} \mapsto \boldsymbol{y}(\boldsymbol{x})$ is a homeomorphism ([Ked10, 1.3.3]). Hence, there exists an open neighborhood $V \subset R^n$ of $\boldsymbol{y}(\boldsymbol{x}^{(0)})$ such that $w(\boldsymbol{y})$ is a generic cyclic vector of M^{alg} for all $\boldsymbol{y} \in V$. Since $\Gamma^{\mathrm{alg}}_{\mathrm{con}}$ is dense in Γ^{alg}, $w(\boldsymbol{y}) \in M$ for $\boldsymbol{y} \in V \cap (\Gamma^{\mathrm{alg}}_{\mathrm{con}}[p^{-1}])^n \neq \phi$ is a generic cyclic vector of M^{alg}. $\square$

6 Frobenius equation and log-growth

In [CT09, § 7.2], Chiarellotto and Tsuzuki compute the log-growth of a solution y of a Frobenius equation

$$ay + by^{\sigma} + cy^{\sigma^2} = 0,\ a, b, c \in K[\![x]\!]_0.$$

In [Nak13], Nakagawa proves a generalization of Chiarellotto-Tsuzuki's result for a Frobenius equation

$$a_0 y + a_1 y^{\sigma} + \cdots + a_n y^{\sigma^n} = 0,\ a_i \in \mathcal{E}^{\dagger}$$

under the assumption that the number of breaks of the Newton polygon of $a_0 + a_1\sigma + \cdots + a_n\sigma^n$ is equal to n. We generalize Nakagawa's result without any assumption on the Newton polygon:

Theorem 6.1 (A generalization of Nakagawa's theorem ([Nak13, 1.1]))**.** *Let*

$$f(\sigma) = a_0 + a_1\sigma + \cdots + a_n\sigma^n \in \Gamma^{\mathrm{alg}}_{\mathrm{con}}[p^{-1}]\{\sigma\},\ a_0 \neq 0,\ a_n \neq 0,\ n \geq 1$$

be a twisted polynomial satisfying the condition (*) in Definition 5.1 (ii) with slopes $-s_1 < \cdots < -s_k$. If $y \in \Gamma^{\mathrm{alg}}_{\mathrm{log,an,con}}$ is a solution of the Frobenius equation

$$f(\sigma)y = a_0 y + a_1 y^{\sigma} + \cdots + a_n y^{\sigma^n} = 0, \tag{1}$$

then y is either bounded or exactly of log-growth s_j for some j such that $s_j > 0$.

Remark 6.2. For a (σ, ∇)-module M of rank n over $\Gamma_{\mathrm{con}}[p^{-1}]$, we construct a Frobenius equation $f(\sigma)y = 0$ satisfying the assumption of Theorem 6.1 (see Construction 7.1). The ambiguity of the log-growth of y in Theorem 6.1 is owing to the fact that $M_{\mathcal{E}}/M^0_{\mathcal{E}}$ may not be pure as a σ-module. One can expect that if M is PBQ, then y is exactly of log-growth s_1, as is the case for $n = 2$ ([CT09, 7.3]).

We divide the proof into two parts: the first part is an estimation of an upper bound of the log-growth of y (easier), and the second part is an estimation of a lower bound of the log-growth of y (harder). The condition (*) will be used only in the second part. The integer j in Theorem 6.1 will be determined in § 6.3.

Notation 6.3. In this section, we keep the notation in Theorem 6.1. Let $0 = i(0) < i(1) < \cdots < i(k) = n$ be the x-coordinates of the vertices of $\mathrm{NP}(f(\sigma))$. By Lemma 4.4, there exists a real number ρ_0 sufficiently close to 1 from the left such that for all $i \in \{0, 1, \dots, n\}$, we have

$$a_i \in \Gamma^{\mathrm{alg}}_r,\ y \in \Gamma^{\mathrm{alg}}_{\mathrm{log,an},r},$$

where $r = -\log_p \rho_0^{q^n}$ and

$$|a_i|_0(\rho) = \rho^{\alpha(i)}|a_i|_0(1)\ \forall \rho \in [\rho_0^{q^n}, 1)$$

for some $\alpha(i) \in \mathbb{Q}$; we fix such a ρ_0.

6.1 Estimation of upper bound

Proposition 6.4 (A refinement of [CT09, 6.12]). *Let $j \in \{0, \dots, k-1\}$. We assume*

$$\sup_{i(j-1)\le i\le i(j)} |a_i y^{\sigma^i}|_0(\rho) \le \sup_{i(j)+1\le i\le n} |a_i y^{\sigma^i}|_0(\rho) \ \forall \rho \in [\rho_0, 1); \tag{2}$$

when $j = 0$, we set $\sup_{i(j-1)\le i\le i(j)} |a_i y^{\sigma^i}|_0(\rho) = |a_0 y|_0(\rho)$. Then, we have

(i) For any $\rho \in [\rho_0, 1)$ and $m \in \mathbb{N}$, there exist an integer $N \in \{0, \dots, n-1\}$, which depends only on m, and a sequence ε_{iu} of integers, which depends on ρ and m, defined for

$$I_m := \{(i, u) \in \mathbb{Z}^2; i(j) + 1 \le i \le n,\ -m - i(j) \le u \le 0\}$$

satisfying the following conditions:

(a)

$$\log |y|_0(\rho^{q^{-m}}) - \log |y|_0(\rho^{q^{-N}}) \le \sum_{(i,u)\in I_m} \varepsilon_{iu} \log |a_i/a_{i(j)}|_0(\rho^{q^u});$$

(b) $\varepsilon_{iu} \in \{0, 1\}$ and

$$\sum_{(i,u)\in I_m} (i - i(j))\varepsilon_{iu} = m - N.$$

(ii) y has log-growth s_{j+1}.

Proof. (i) We fix ρ and proceed by induction on m. When $m \le n-1$, we set $N = m$ and $\varepsilon_{iu} \equiv 0$ for all $(i, u) \in I_m$. Then, we have nothing to prove. Assume that the assertion is true for the integers less than or equal to $m - 1$ with $m \ge n$. By (2) for $\rho = \rho^{q^{-m-i(j)}}$, we have

$$|a_{i(j)} y^{\sigma^{i(j)}}|_0(\rho^{q^{-m-i(j)}}) \le \sup_{i(j)+1\le i\le n} |a_i y^{\sigma^i}|_0(\rho^{q^{-m-i(j)}}). \tag{3}$$

We choose $i' \in \{i(j) + 1, \dots, n\}$ such that

$$\sup_{i(j)+1\le i\le n} |a_i y^{\sigma^i}|_0(\rho^{q^{-m-i(j)}}) = |a_{i'} y^{\sigma^{i'}}|_0(\rho^{q^{-m-i(j)}}). \tag{4}$$

Recall $|\sigma(\cdot)|_0(\eta) = |\cdot|_0(\eta^q)$ for all $\eta \in (0, 1)$ (§ 4.1). Then, by (3) and (4),

$$\log |y|_0(\rho^{q^{-m}}) - \log |y|_0(\rho^{q^{-m-i(j)+i'}}) \le \log |a_{i'}/a_{i(j)}|_0(\rho^{q^{-m-i(j)}}). \tag{5}$$

By the induction hypothesis for $m + i(j) - i'$, there exist an integer $N \in \{0, \dots, n-1\}$ and a sequence ε'_{iu} of 0 or 1 defined for $I_{m+i(j)-i'}$ such that

$$\log |y|_0(\rho^{q^{-m-i(j)+i'}}) - \log |y|_0(\rho^{q^{-N}}) \le \sum_{(i,u)\in I_{m+i(j)-i'}} \varepsilon'_{iu} \log |a_i/a_{i(j)}|_0(\rho^{q^u}), \tag{6}$$

$$\sum_{(i,u)\in I_{m+i(j)-i'}} (i - i(j))\varepsilon'_{iu} = m + i(j) - i' - N. \tag{7}$$

For $(i, u) \in I_m$, we define

$$\varepsilon_{iu} := \begin{cases} \varepsilon'_{iu} & \text{if } (i, u) \in I_{m+i(j)-i'} \\ 1 & \text{if } (i, u) = (i', -m - i(j)) \\ 0 & \text{otherwise.} \end{cases}$$

Then, by adding (5) to (6), the inequality in (a) follows. The condition $\varepsilon_{iu} \in \{0, 1\}$ follows by construction, and the equality in (b) follows from (7).

(ii) We fix $\rho \in [\rho_0, 1)$ for a while. Let m be a natural number such that $\rho^{q^m} \in [\rho_0, \rho_0^{q^{-1}})$. By applying (i) to $(\rho, m) = (\rho^{q^m}, m)$, there exist an integer $N(m) \in \{0, \ldots, n-1\}$ and a sequence $\varepsilon_{iu}^{(m)}$ of 0 or 1 defined for $(i, u) \in I_m$ such that

$$\log |y|_0(\rho) - \log |y|_0(\rho^{q^{m-N(m)}}) \leq \sum_{(i,u)\in I_m} \varepsilon_{iu}^{(m)} \log |a_i/a_{i(j)}|_0(\rho^{q^{u+m}}) \tag{8}$$

$$\sum_{(i,u)\in I_m} (i - i(j))\varepsilon_{iu}^{(m)} = m - N(m). \tag{9}$$

For $i(j) + 1 \leq i \leq n$, there exists $v(i) \in \mathbb{Q}$ such that

$$|a_i/a_{i(j)}|_0(\eta) = \eta^{v(i)}|a_i/a_{i(j)}|_0(1) \ \forall \eta \in [\rho_0, 1) \tag{10}$$

by Notation 6.3. Moreover,

$$-\frac{1}{i - i(j)} \log_q |a_i/a_{i(j)}|_0(1) \geq -s_{j+1} \tag{11}$$

by the convexity of the Newton polygon of $f(\sigma)$. By (10) and (11),

$$\text{RHS of (8)} \leq \sum_{(i,u)\in I_m} \varepsilon_{iu}^{(m)} q^{u+m} v(i) \log \rho + \sum_{(i,u)\in I_m} (i - i(j))\varepsilon_{iu}^{(m)} s_{j+1} \log q. \tag{12}$$

Let $v := \max\{\pm v(i); i(j) + 1 \leq i \leq n\}$. Then, the first sum in RHS of (12) is bounded above by

$$\sum_{(i,u)\in I_m} \varepsilon_{iu}^{(m)} q^{u+m} v \log (1/\rho) \leq \sum_{(i,u)\in I_m} q^{u+m} v \log (1/\rho) = (n - i(j))\frac{q^m - q^{-i(j)-1}}{1 - q^{-1}} v \log (1/\rho)$$
$$\leq (n - i(j))\frac{q^{m+1}}{q-1} v \log (1/\rho) = (n - i(j))\frac{q}{q-1} v \log (1/\rho^{q^m}) \leq n\frac{q}{q-1} v \log (1/\rho_0).$$

By (9), the second sum in RHS of (12) is equal to

$$(m - N(m))s_{j+1} \log q.$$

Thus, (8) leads to

$$\begin{aligned} |y|_0(\rho) &\leq C|y|_0(\rho^{q^{m-N(m)}}) \cdot q^{(m-N(m))s_{j+1}} \\ &= C|y|_0(\rho^{q^{m-N(m)}}) \cdot (\log (1/\rho^{q^{m-N(m)}}))^{s_{j+1}} \cdot (\log (1/\rho))^{-s_{j+1}}, \end{aligned} \tag{13}$$

where $C := \exp\{nq(q-1)^{-1} v \log (1/\rho_0)\}$ is a constant independent of ρ. Since $\rho^{q^{m-N(m)}} \in [\rho_0, \rho_0^{q^{-n}})$, the functions $|y|_0(\rho^{q^{m-N(m)}})$ and $(\log (1/\rho^{q^{m-N(m)}}))^{s_{j+1}}$ are bounded when ρ runs over $[\rho_0, 1)$: note that the function $[\rho_0, 1) \to \mathbb{R}; \rho \mapsto |y|_0(\rho)$ is continuous. Thus, (13) implies the desired estimation

$$|y|_0(\rho) = O((\log (1/\rho))^{-s_{j+1}}) \text{ as } \rho \uparrow 1.$$

□

6.2 Estimation of lower bound

We start with converting the condition (*) into the lemma:

Lemma 6.5. *For any $j \in \{0, \ldots, k-1\}$, $i \in \{i(j+1)+1, \ldots, n\}$, and $i' \in \{0, \ldots, i(j+1)-1\}$, we have*

$$\log |a_{i'-i(j+1)+i}|_0(1) - \log |a_{i'}|_0(1) > \log |a_i|_0(1) - \log |a_{i(j+1)}|_0(1).$$

Proof. For $0 \leq i \leq n$, we denote by P_i the point $(i, -\log_q |a_i|_0(1))$. We also denote by L_1 and L_2 the segments $P_{i'}P_{i'-i(j+1)+i}$ and $P_{i(j+1)}P_i$, respectively. Let a and b be the slopes of L_1 and L_2, respectively. We have only to prove $a < b$. Let us consider separately the cases where $i' - i(j+1) + i \leq i(j+1)$ or $i' - i(j+1) + i > i(j+1)$. In the first case, we have $a \leq -s_{j+1}$ by the condition (*). By the convexity of the Newton polygon of $f(\sigma)$, we have $-s_{j+1} < b$. Hence, $a < b$. In the latter case, the segment L_1 intersects with L_2. Since $P_{i(j+1)}$ is under L_1, we have $a < b$. $\square$

Notation 6.6. By Lemmas 4.4 and 6.5, after choosing ρ_0 sufficiently large if necessary, we may assume the following condition: for any $j \in \{0, \ldots, k-1\}$, $i \in \{i(j+1)+1, \ldots, n\}$, and $i' \in \{0, \ldots, i(j+1)-1\}$, we have

$$\log |a_{i'-i(j+1)+i}|_0(\rho_2) - \log |a_{i'}|_0(\rho_2) > \log |a_i|_0(\rho_3) - \log |a_{i(j+1)}|_0(\rho_3) \ \forall \rho_2, \rho_3 \in [\rho_0, 1); \qquad (14)$$

indeed, both sides of the inequality are continuous with respect to ρ_2 and ρ_3, respectively, and converge to $\log |a_{i'-i(j+1)+i}|_0(1) - \log |a_{i'}|_0(1)$ and $\log |a_i|_0(1) - \log |a_{i(j+1)}|_0(1)$ as $\rho_2, \rho_3 \uparrow 1$, respectively.

To estimate the function $|y|_0(\rho)$ of ρ from below, we need to combine several inequalities which are similar to inequality (2).

Assumption 6.7. Let $j \in \{0, \ldots, k-1\}$. In the rest of this subsection, we assume the following:

$$\sup_{i(0) \leq i \leq i(1)} |a_i y^{\sigma^i}|_0(\rho) \leq \sup_{i(1)+1 \leq i \leq n} |a_i y^{\sigma^i}|_0(\rho) \ \forall \rho \in [\rho_0^{q^{-n}}, 1),$$

$$\sup_{i(1) \leq i \leq i(2)} |a_i y^{\sigma^i}|_0(\rho) \leq \sup_{i(2)+1 \leq i \leq n} |a_i y^{\sigma^i}|_0(\rho) \ \forall \rho \in [\rho_0^{q^{-n}}, 1),$$

$$\vdots$$

$$\sup_{i(j-1) \leq i \leq i(j)} |a_i y^{\sigma^i}|_0(\rho) \leq \sup_{i(j)+1 \leq i \leq n} |a_i y^{\sigma^i}|_0(\rho) \ \forall \rho \in [\rho_0^{q^{-n}}, 1);$$

when $j = 0$, we set $\sup_{i(j-1) \leq i \leq i(j)} |a_i y^{\sigma^i}|_0(\rho) := |a_0 y|_0(\rho)$. We also assume

$$\sup_{i(j) \leq i \leq i(j+1)} |a_i y^{\sigma^i}|_0(\rho) > \sup_{i(j+1)+1 \leq i \leq n} |a_i y^{\sigma^i}|_0(\rho) \ \exists \rho \in [\rho_0^{q^{-n}}, 1);$$

when $j = k-1$, we set $\sup_{i(j+1)+1 \leq i \leq n} |a_i y^{\sigma^i}|_0(\rho) := 0$.

Lemma 6.8. *Assume that* $\rho_1 \in [\rho_0^{q^{-n}}, 1)$ *satisfies*

$$\sup_{i(j) \leq i \leq i(j+1)} |a_i y^{\sigma^i}|_0(\rho_1) > \sup_{i(j+1)+1 \leq i \leq n} |a_i y^{\sigma^i}|_0(\rho_1).$$

(i) We have

$$\sup_{0 \leq i \leq i(j+1)-1} |a_i y^{\sigma^i}|_0(\rho_1) \geq |a_{i(j+1)} y^{\sigma^{i(j+1)}}|_0(\rho_1).$$

(ii) Let $i' \in \{0, \ldots, i(j+1)-1\}$ *be an integer such that*

$$|a_{i'} y^{\sigma^{i'}}|_0(\rho_1) = \sup_{0 \leq i \leq i(j+1)-1} |a_i y^{\sigma^i}|_0(\rho_1).$$

Then, we have

$$\sup_{i(j) \leq i \leq i(j+1)} |a_i y^{\sigma^i}|_0(\rho_1^{q^{i'-i(j+1)}}) > \sup_{i(j+1)+1 \leq i \leq n} |a_i y^{\sigma^i}|_0(\rho_1^{q^{i'-i(j+1)}}).$$

Proof. (i) Suppose the contrary. Then, we have $|a_{i(j+1)} y^{\sigma^{i(j+1)}}|_0(\rho_1) > \sup_{i \neq i(j+1)} |a_i y^{\sigma^i}|_0(\rho_1) \geq 0$ by the assumption of the lemma. By (1), we have $a_{i(j+1)} y^{\sigma^{i(j+1)}} = -\sum_{i \neq i(j+1)} a_i y^{\sigma^i}$. By taking $|\cdot|_0(\rho_1)$, we have $|a_{i(j+1)} y^{\sigma^{i(j+1)}}|_0(\rho_1) \leq \sup_{i \neq i(j+1)} |a_i y^{\sigma^i}|_0(\rho_1)$, which is a contradiction.

(ii) By (i) and the assumption of the lemma, we have

$$|a_{i'}y^{\sigma^{i'}}|_0(\rho_1) = \sup_{0\leq i\leq n} |a_i y^{\sigma^i}|_0(\rho_1). \tag{15}$$

For $i \in \{i(j+1)+1, \ldots, n\}$, we have

$$\begin{aligned} &\log|y^{\sigma^{i(j+1)}}|_0(\rho_1^{q^{i'-i(j+1)}}) - \log|y^{\sigma^i}|_0(\rho_1^{q^{i'-i(j+1)}}) = \log|y|_0(\rho_1^{q^{i'}}) - \log|y|_0(\rho_1^{q^{i'-i(j+1)+i}}) \\ =&\log|a_{i'}y^{\sigma^{i'}}|_0(\rho_1) - \log|a_{i'-i(j+1)+i}y^{\sigma^{i'-i(j+1)+i}}|_0(\rho_1) - \log|a_{i'}|_0(\rho_1) + \log|a_{i'-i(j+1)+i}|_0(\rho_1) \\ \geq&\log|a_{i'-i(j+1)+i}|_0(\rho_1) - \log|a_{i'}|_0(\rho_1) > \log|a_i|_0(\rho_1^{q^{i'-i(j+1)}}) - \log|a_{i(j+1)}|_0(\rho_1^{q^{i'-i(j+1)}}), \end{aligned}$$

where the first and second inequalities follow from (15) and (14), respectively. Thus, we obtain

$$|a_{i(j+1)}y^{\sigma^{i(j+1)}}|_0(\rho_1^{q^{i'-i(j+1)}}) > \sup_{i(j+1)+1\leq i\leq n} |a_i y^{\sigma^i}|_0(\rho_1^{q^{i'-i(j+1)}}),$$

which implies the assertion.

□

Construction 6.9. Fix $\rho_1 \in [\rho_0^{q^{-n}}, 1)$ such that

$$\sup_{i(j)\leq i\leq i(j+1)} |a_i y^{\sigma^i}|_0(\rho_1) > \sup_{i(j+1)+1\leq i\leq n} |a_i y^{\sigma^i}|_0(\rho_1).$$

By induction on $l \in \mathbb{N}$, we construct a strictly decreasing sequence $\{m(l)\}_l$ of integers less than or equal to $i(j+1)$, and a sequence $\varepsilon_{iu}^{(l)}$ of integers defined for

$$\mathcal{I}_l := \{(i,u) \in \mathbb{Z}^2; 0 \leq i \leq i(j+1)-1, m(l)-i(j+1) \leq u \leq 0\}$$

satisfying the following conditions:

(a)
$$\log|y|_0(\rho_1^{q^{m(l)}}) - \log|y|_0(\rho_1^{q^{i(j+1)}}) \geq \sum_{(i,u)\in\mathcal{I}_l} \varepsilon_{iu}^{(l)} \log|a_{i(j+1)}/a_i|_0(\rho_1^{q^u}).$$

(b) $\varepsilon_{iu}^{(l)} \in \{0,1\}$ and
$$\sum_{(i,u)\in\mathcal{I}_l} (i(j+1)-i)\varepsilon_{iu}^{(l)} = i(j+1) - m(l).$$

(c)
$$\sup_{i(j)\leq i\leq i(j+1)} |a_i y^{\sigma^i}|_0(\rho_1^{q^{m(l)-i(j+1)}}) > \sup_{i(j+1)+1\leq i\leq n} |a_i y^{\sigma^i}|_0(\rho_1^{q^{m(l)-i(j+1)}}).$$

We set $m(0) := i'$ where i' is defined in Lemma 6.8 (ii), and define

$$\varepsilon_{iu}^{(0)} := \begin{cases} 1 & \text{if } (i,u) = (i',0) \\ 0 & \text{otherwise.} \end{cases}$$

Since $|a_{i'}y^{\sigma^{i'}}|_0(\rho_1) \geq |a_{i(j+1)}y^{\sigma^{i(j+1)}}|_0(\rho_1)$ by Lemma 6.8 (i), condition (a) follows. Condition (b) follows by definition. Condition (c) follows from Lemma 6.8 (ii).

Given $m(l)$ and $\varepsilon_{iu}^{(l)}$, we can apply Lemma 6.8 to $\rho_1 = \rho_1^{q^{m(l)-i(j+1)}}$ by condition (c) for $m(l)$: let $i' \in \{0, \ldots, i(j+1)-1\}$ be the integer defined in Lemma 6.8 (ii). Since $|a_{i'}y^{\sigma^{i'}}|_0(\rho_1^{q^{m(l)-i(j+1)}}) \geq |a_{i(j+1)}y^{\sigma^{i(j+1)}}|_0(\rho_1^{q^{m(l)-i(j+1)}})$ by Lemma 6.8 (i), we have

$$\log|y|_0(\rho_1^{q^{m(l)-i(j+1)+i'}}) - \log|y|_0(\rho_1^{q^{m(l)}}) \geq \log|a_{i(j+1)}/a_{i'}|_0(\rho_1^{q^{m(l)-i(j+1)}}). \tag{16}$$

We set $m(l+1) := m(l) - i(j+1) + i' < m(l)$ and define $\varepsilon_{iu}^{(l+1)}$ for $(i,u) \in \mathcal{I}_{l+1}$ by

$$\varepsilon_{iu}^{(l+1)} := \begin{cases} \varepsilon_{iu}^{(l)} & \text{if } (i,u) \in \mathcal{I}_l \\ 1 & \text{if } (i,u) = (i', m(l+1) - i(j+1)) \\ 0 & \text{otherwise.} \end{cases}$$

We verify the conditions (a), (b), and (c). By adding (16) to the inequality in (a) for $m(l)$, condition (a) follows. Condition (b) follows by construction. Condition (c) follows from Lemma 6.8 (ii).

Proposition 6.10. *If y is non-zero and has log-growth $\alpha \in \mathbb{R}_{\geq 0}$, then $\alpha \geq s_{j+1}$.*

Proof. Obviously, we may assume $s_{j+1} > 0$. For $i \in \{0, \dots, i(j+1) - 1\}$, there exists $v(i) \in \mathbb{Q}$ such that

$$|a_{i(j+1)}/a_i|_0(\rho) = \rho^{v(i)} |a_{i(j+1)}/a_i|_0(1) \ \forall \rho \in [\rho_0^{q^n}, 1). \tag{17}$$

Moreover,

$$-\frac{1}{i(j+1) - i} \log_q |a_{i(j+1)}/a_i|_0(1) \leq -s_{j+1} \tag{18}$$

by the convexity of the Newton polygon of $f(\sigma)$.

We retain the notation in Construction 6.9. By (17), (18), and the inequality (a) for $m(l)$, we have

$$\log |y|_0(\rho_1^{q^{m(l)}}) - \log |y|_0(\rho_1^{q^{i(j+1)}}) \geq \sum_{(i,u) \in \mathcal{I}_l} \varepsilon_{iu}^{(l)} q^u v(i) \log \rho_1 + \sum_{(i,u) \in \mathcal{I}_l} (i(j+1) - i) \varepsilon_{iu}^{(l)} s_{j+1} \log q. \tag{19}$$

Let $v := \inf\{\pm v(i); 0 \leq i \leq i(j+1) - 1\}$. Then, the first sum in RHS of (19) is bounded below by

$$\sum_{(i,u) \in \mathcal{I}_l} \varepsilon_{iu}^{(l)} q^u v \log(1/\rho_1) \geq \sum_{(i,u) \in \mathcal{I}_l} q^u v \log(1/\rho_1) = i(j+1) \frac{1 - q^{m(l) - i(j+1) - 1}}{1 - q^{-1}} v \log(1/\rho_1) \geq n \frac{q}{q-1} v \log(1/\rho_1).$$

By condition (b) for $m(l)$, the second sum in RHS of (19) is equal to

$$(i(j+1) - m(l)) s_{j+1} \log q.$$

Therefore, (19) leads to

$$|y|_0(\rho_1^{q^{m(l)}}) \geq C |y|_0(\rho_1^{q^{i(j+1)}}) \cdot q^{(i(j+1) - m(l)) s_{j+1}} = C |y|_0(\rho_1^{q^{i(j+1)}}) \cdot q^{i(j+1) s_{j+1}} (\log(1/\rho_1))^{s_{j+1}} \cdot (\log(1/\rho_1^{q^{m(l)}}))^{-s_{j+1}}, \tag{20}$$

where $C := \exp\{nq(q-1)^{-1} v \log(1/\rho_1)\}$. Note that

$$C |y|_0(\rho_1^{q^{i(j+1)}}) \cdot q^{i(j+1) s_{j+1}} (\log(1/\rho_1))^{s_{j+1}}$$

is a positive constant independent of l. Since $m(l) \to -\infty$ as $l \to \infty$, (20) implies

$$|y|_0(\rho) \neq O((\log(1/\rho))^{-\beta}) \text{ as } \rho \uparrow 1$$

for any $\beta \in \mathbb{R}_{< s_{j+1}}$. In other words, y does not have log-growth strictly less than s_{j+1}. Hence, $\alpha \geq s_{j+1}$. ∎

6.3 Proof of Theorem 6.1

Let ρ_0 be as in Notation 6.6. For $j \in \{0, 1, \dots, k-2\}$, we consider the following condition on y:

$$(C_j): \sup_{i(j) \leq i \leq i(j+1)} |a_i y^{\sigma^i}|_0(\rho) \leq \sup_{i(j+1)+1 \leq i \leq n} |a_i y^{\sigma^i}|_0(\rho) \ \forall \rho \in [\rho_0, 1).$$

Let $j \in \{0, \dots, k-2\}$ be the least integer such that condition (C_j) does not hold; if condition (C_j) holds for all j, then we set $j = k-1$. Then, j satisfies the assumption in Proposition 6.4; when $j = 0$, the assumption follows from (1). In addition, j satisfies Assumption 6.7; when $j = k-1$, the assumption follows from $y \neq 0$. Therefore, the assertion follows from Propositions 6.4 and 6.10.

7 Proof of Theorem 4.19

In this section, we assume that k_K is algebraically closed as in Assumption 4.16. For a (σ, ∇)-module over $K[\![x]\!]_0$, Chiarellotto and Tsuzuki define a Frobenius equation ([CT09, Proof of Theorem 6.17 (2)]). Then, they interpret their conjecture $\mathbf{LGF}_{K[\![x]\!]_0}$ as a problem on the Frobenius equation. For a (σ, ∇)-module over $\Gamma_{\mathrm{con}}[p^{-1}]$, their method can be applied as follows.

Construction 7.1. Set $R := \Gamma_{\mathrm{con}}[p^{-1}]$. Let M be a (σ, ∇)-module of rank n over R solvable in $\Gamma_{\log,\mathrm{an},\mathrm{con}}$, and $s_1 < \cdots < s_k$ the generic Frobenius slopes of M. Assume $q^{s_j} \in \mathbb{Q}$ for all j. We choose a generic cyclic vector e of $\Gamma^{\mathrm{alg}}_{\mathrm{con}}[p^{-1}] \otimes_R M^\vee$ by Theorem 5.2, and we write

$$\varphi^n(e) = -(a_{n-1}\varphi^{n-1}(e) + \cdots + a_0 e),\ a_i \in \Gamma^{\mathrm{alg}}_{\mathrm{con}}[p^{-1}].$$

Let v be an element of $\mathfrak{Sol}(M)$ such that $\varphi(v) = \gamma v$ for some $\gamma \in (\Gamma^{\mathrm{alg}}_{\mathrm{con}}[p^{-1}])^\times$. By identifying $\mathfrak{Sol}(M)$ as a submodule of $\Gamma^{\mathrm{alg}}_{\log,\mathrm{an},\mathrm{con}} \otimes_R M^\vee$, we write

$$v = y_0 e + y_1 \varphi(e) + \cdots + y_{n-1}\varphi^{n-1}(e),\ y_i \in \Gamma^{\mathrm{alg}}_{\log,\mathrm{an},\mathrm{con}}.$$

Then, we obtain the relation

$$\begin{pmatrix} & & & -a_0 \\ 1 & & & -a_1 \\ & \ddots & & \vdots \\ & & 1 & -a_{n-1} \end{pmatrix} \sigma \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix} = \gamma \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}. \tag{21}$$

By elimination, $y := y_{n-1}$ satisfies the following Frobenius equation:

$$y = -\sum_{0 \le i \le n-1} \frac{\sigma^i(a_{n-i-1})}{\gamma\sigma(\gamma)\ldots\sigma^i(\gamma)} \sigma^{i+1}(y). \tag{22}$$

Note that the slopes of the twisted polynomial

$$1 + \frac{a_{n-1}}{\gamma}\sigma + \cdots + \frac{\sigma^{n-1}(a_0)}{\gamma\sigma(\gamma)\cdots\sigma^{n-1}(\gamma)}\sigma^n$$

are $-s_k - s < \cdots < -s_1 - s$, where $s = -\log_q |\gamma|$.

Lemma 7.2. *We retain the notation in Construction 7.1.*

(i) For $\lambda \in \mathbb{R}$, we have $v \in \mathfrak{Sol}_\lambda(M)$ if and only if $y \in \mathrm{Fil}_\lambda \Gamma^{\mathrm{alg}}_{\log,\mathrm{an},\mathrm{con}}$.

(ii) We have either $v \in \mathfrak{Sol}_0(M)$ or $v \in \mathfrak{Sol}_{s+s_j}(M) \setminus \mathfrak{Sol}_{(s+s_j)-}(M)$ for some j such that $s + s_j > 0$.

Proof. (i) Since we have

$$\mathfrak{Sol}_\lambda(M) \subset \mathrm{Fil}_\lambda \Gamma_{\log,\mathrm{an},\mathrm{con}} \otimes_R M^\vee \subset \mathrm{Fil}_\lambda \Gamma^{\mathrm{alg}}_{\log,\mathrm{an},\mathrm{con}} \otimes_R M^\vee \cong \mathrm{Fil}_\lambda \Gamma^{\mathrm{alg}}_{\log,\mathrm{an},\mathrm{con}} \otimes_{\Gamma^{\mathrm{alg}}_{\mathrm{con}}[p^{-1}]} \Gamma^{\mathrm{alg}}_{\mathrm{con}}[p^{-1}] \otimes_R M^\vee,$$

$v \in \mathfrak{Sol}_\lambda(M)$ implies $y \in \mathrm{Fil}_\lambda \Gamma^{\mathrm{alg}}_{\log,\mathrm{an},\mathrm{con}}$. Assume $y \in \mathrm{Fil}_\lambda \Gamma^{\mathrm{alg}}_{\log,\mathrm{an},\mathrm{con}}$. By (21) and Lemma 4.12, we have $y_i \in \mathrm{Fil}_\lambda \Gamma^{\mathrm{alg}}_{\log,\mathrm{an},\mathrm{con}}$ by decreasing induction on i. Hence, $v \in \mathrm{Fil}_\lambda \Gamma^{\mathrm{alg}}_{\log,\mathrm{an},\mathrm{con}} \otimes_R M^\vee$. Since $\mathrm{Fil}_\lambda \Gamma^{\mathrm{alg}}_{\log,\mathrm{an},\mathrm{con}} \cap \Gamma_{\log,\mathrm{an},\mathrm{con}} = \mathrm{Fil}_\lambda \Gamma_{\log,\mathrm{an},\mathrm{con}}$ by definition, we have $v \in \mathrm{Fil}_\lambda \Gamma_{\log,\mathrm{an},\mathrm{con}} \otimes_R M^\vee$, i.e., $v \in \mathfrak{Sol}_\lambda(M)$.

(ii) The assertion follows from (i) and Theorem 6.1.

□

We deduce Proposition 4.18 and Theorem 4.19 from Lemma 7.2 (ii) and the following lemma.

Lemma 7.3. *Let M be a (σ, ∇)-module over $\mathcal{E}$ and $\lambda_{\max}$ the highest Frobenius slope of M. If M is PBQ, then $(M^\vee)_0$ is pure of slope $-\lambda_{\max}$ as a σ-module.*

Proof. We have a canonical isomorphism $(M^\vee)_0 \cong (M/M^0)^\vee$ induced by the canonical paring $M \otimes_{\mathcal{E}} M^\vee \to \mathcal{E}$ (§ 3.2). Hence, $(M^\vee)_0$ is pure as a σ-module by assumption. Moreover, the Frobenius slope λ of $(M^\vee)_0$ is greater than or equal to $-\lambda_{\max}$. Suppose that the assertion is false, i.e., $\lambda > -\lambda_{\max}$. Let M' be the inverse image of $M'' := S_{-\lambda_{\max}}(M^\vee/(M^\vee)_0)$ under the canonical projection $M^\vee \to M^\vee/(M^\vee)_0$. By assumption, $M'' \neq 0$ and there exists a short exact sequence of (σ, ∇)-modules over $\mathcal{E}$:

$$0 \longrightarrow (M^\vee)_0 \longrightarrow M' \longrightarrow M'' \longrightarrow 0.$$

By [CT11, 4.2], the above exact sequence splits as a sequence of (σ, ∇)-modules. Since $M''_0 \neq 0$, we have $(M^\vee)_0 \subsetneq M'_0 \subset (M^\vee)_0$, which is a contradiction. □

Proof of Proposition 4.18. By replacing (M, φ, ∇) by (M, φ^h, ∇) for sufficiently large $h \in \mathbb{N}$, we may assume $q^s \in \mathbb{Q}$ for all special and generic Frobenius slopes s of M. Then, we may apply Construction 7.1 to M. We retain the notation in Construction 7.1. Note that $s_k = \lambda_{\max}$ by definition.

(i) Let $v \in \mathfrak{Sol}(M)$ be a non-zero Frobenius eigenvector of slope s. By Grothendieck-Katz specialization theorem ([Ked10, 15.3.2]), we have $s \geq -s_k$. By Lemma 7.2 (ii), we have $v \in \mathfrak{Sol}_{s+s_k}(M)$. Hence, we have $S_{\lambda-\lambda_{\max}}(\mathfrak{Sol}(M)) \subset \mathfrak{Sol}_\lambda(M)$ for all $\lambda \in \mathbb{R}$. By taking $(\cdot)^\perp$ with respect to the canonical pairing $\mathfrak{V}(M) \otimes_K \mathfrak{Sol}(M) \to K$, we obtain $(S_{\lambda-\lambda_{\max}}(\mathfrak{V}(M^\vee)))^\perp \supset \mathfrak{V}(M)^\lambda$.

(ii) By (i), we have only to prove $(S_{-\lambda_{\max}}(\mathfrak{V}(M^\vee)))^\perp \subset \mathfrak{V}(M)^0$. Since $\mathfrak{Sol}_0(M) = (M^\vee)^{\nabla=0}$, we have $\mathfrak{Sol}_0(M) \subset (M^\vee_{\mathcal{E}})_0$ by the characterization of $(M^\vee_{\mathcal{E}})_0$. By Lemma 7.3, $(M^\vee_{\mathcal{E}})_0$, and hence, $\mathfrak{Sol}_0(M)$ are pure of slope $-\lambda_{\max}$ as a σ-module, i.e., $\mathfrak{Sol}_0(M) \subset S_{-\lambda_{\max}}(\mathfrak{Sol}(M))$. By taking $(\cdot)^\perp$ with respect to the canonical pairing $\mathfrak{V}(M) \otimes_K \mathfrak{Sol}(M) \to K$, we obtain the assertion.

□

Proof of Theorem 4.19. Similarly to the proof of Proposition 4.18, we may apply Construction 7.1 to M again.

(i) By the definition of $\mathfrak{V}(M)^\bullet$, we have only to prove that the filtration $\mathfrak{Sol}_\bullet(M)$ is rational and right continuous.

We first prove the rationality of breaks λ of $\mathfrak{Sol}_\bullet(M)$. We may assume $\lambda > 0$. Since $\mathfrak{Sol}_{\lambda-}(M)$ is a direct summand of $\mathfrak{Sol}_{\lambda+}(M)$ as a σ-module, we can choose a Frobenius eigenvector $v \in \mathfrak{Sol}_{\lambda+}(M) \setminus \mathfrak{Sol}_{\lambda-}(M)$ of slope s. By $v \notin \mathfrak{Sol}_0(M)$ and Lemma 7.2 (ii), we have $v \in \mathfrak{Sol}_{s+s_j}(M) \setminus \mathfrak{Sol}_{(s+s_j)-}(M)$ for some j such that $s + s_j > 0$, i.e., $\lambda = s + s_j \in \mathbb{Q}$.

We prove the right continuity of $\mathfrak{Sol}_\bullet(M)$. Suppose the contrary, i.e., there exists $\lambda \in \mathbb{R}_{\geq 0}$ such that $\mathfrak{Sol}_\lambda(M) \neq \mathfrak{Sol}_{\lambda+}(M)$. Let $\Delta(M)$ be the set of rational numbers consisting of 0 and $s+s_j$ where s is a Frobenius slope of $\mathfrak{Sol}(M)$. Fix $\lambda' \in \mathbb{R}_{>\lambda}$ sufficiently close to λ such that $\mathfrak{Sol}_{\lambda+}(M) = \mathfrak{Sol}_{\lambda'}(M)$ and $\Delta(M) \cap (\lambda, \lambda'] = \phi$. Since $\mathfrak{Sol}_\lambda(M)$ is a direct summand of $\mathfrak{Sol}_{\lambda+}(M)$ as a σ-module, we can choose a Frobenius eigenvector $v \in \mathfrak{Sol}_{\lambda+}(M) \setminus \mathfrak{Sol}_\lambda(M)$ of slope s. By Lemma 7.2 (ii), we have either $v \in \mathfrak{Sol}_0(M)$ or $v \in \mathfrak{Sol}_{s+s_j}(M) \setminus \mathfrak{Sol}_{(s+s_j)-}(M)$ for some j such that $s + s_j > 0$. In the first case, we have $v \in \mathfrak{Sol}_\lambda(M)$, which is a contradiction. In the latter case, we have $s + s_j > \lambda$ by $v \in \mathfrak{Sol}_{s+s_j}(M)$. Since $v \notin \mathfrak{Sol}_{(s+s_j)-}(M)$, we have $\lambda' \geq s + s_j$. Hence, we have $s + s_j \in \Delta(M) \cap (\lambda, \lambda'] = \phi$, which is a contradiction.

(ii) By Proposition 4.18 (i), we have only to prove $\mathfrak{Sol}_\lambda(M) \subset S_{\lambda-\lambda_{\max}}(\mathfrak{Sol}(M))$ for all $\lambda \geq 0$. Let us consider separately the cases where $k = 1$ or $k = 2$. In the first case, since $\mathfrak{Sol}(M)$ is pure of slope $-\lambda_{\max}$ by Grothendieck-Katz specialization theorem, the assertion is trivial. In the latter case, let $v \in \mathfrak{Sol}_\lambda(M)$ be a non-zero Frobenius eigenvector of slope s. By Grothendieck-Katz specialization theorem, we have $-s_2 \leq s \leq -s_1$. Hence, we have either $v \in \mathfrak{Sol}_0(M)$ or $v \in \mathfrak{Sol}_{s+s_2}(M) \setminus \mathfrak{Sol}_{(s+s_2)-}(M)$ by Lemma 7.2 (ii). In the first case, $v \in \mathfrak{Sol}_0(M) = S_{-s_2}(\mathfrak{Sol}(M)) \subset S_{\lambda-s_2}(\mathfrak{Sol}(M))$ by Proposition 4.18 (ii). In the latter case, we have $s + s_2 \leq \lambda$. Hence, $v \in S_s(\mathfrak{Sol}(M)) \subset S_{\lambda-s_2}(\mathfrak{Sol}(M))$.

□

Let M be a (σ, ∇)-module over $\Gamma_{\mathrm{con}}[p^{-1}]$ solvable in $\Gamma_{\log,\mathrm{an},\mathrm{con}}$. We can expect that any break λ of the special log-growth filtration of M is of the form $-s + \lambda_{\max}$ where s is a special Frobenius slope of M. At this point, as in the proof of Theorem 4.19 (i), we can prove:

Proposition 7.4. *In the above setting, any break λ of the special log-growth filtration of M is of the form $-s+s'$ where s (resp. s') is a special (resp. generic) Frobenius slope of M such that $-s+s' \geq 0$.*

8 Appendix: diagram of rings

For $0 \leq \lambda_1 \leq \lambda_2$, we have the following diagram of rings: all the morphisms are the natural inclusions.

$$\begin{tikzcd}
 & K\llbracket x\rrbracket_0 \arrow[r] \arrow[d] & K\llbracket x\rrbracket_{\lambda_1} \arrow[r] \arrow[d] & K\llbracket x\rrbracket_{\lambda_2} \arrow[r] \arrow[d] & K\{x\} \arrow[r] \arrow[d] & K\llbracket x\rrbracket \\
\Gamma[p^{-1}] \arrow[d, equal] & \Gamma_{\mathrm{con}}[p^{-1}] \arrow[l] \arrow[r] \arrow[d, equal] & \mathrm{Fil}_{\lambda_1}\Gamma_{\mathrm{an,con}} \arrow[r] & \mathrm{Fil}_{\lambda_2}\Gamma_{\mathrm{an,con}} \arrow[r] & \Gamma_{\mathrm{an,con}} \arrow[d, equal] & \\
\mathcal{E} & \mathcal{E}^{\dagger} \arrow[l] \arrow[rrr] & & & \mathcal{R}. &
\end{tikzcd}$$

Acknowledgement

The author is indebted to Professor Nobuo Tsuzuki for drawing the author's attention to Nakagawa's paper [Nak13], and suggesting Remark 6.2. The author thanks the referees for detailed comments. Funding The author is supported by Research Fellowships of Japan Society for the Promotion of Science for Young Scientists.

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